PART B

Unit - VIII

Laplace Transforms - 2

8.1 Introduction

In this unit we discuss **Inverse Laplace Transforms** which can as well be regarded as the reverse process of finding the Laplace transform of a given function. We also discuss *convolution theorem* which helps in finding the inverse Laplace transform. Finally we discuss solution of ordinary differential equations and simultaneous differential equations with a given set of initial conditions referred to as *initial value problems*. This method is highly useful in various branches of engineering. Finally we also present a few applications.

8.2 Inverse Laplace Transforms

We have made a mention of this while defining the Laplace transform of a function $f(t)$. If $L[f(t)] = \overline{f}(s)$ then $f(t)$ is called the **inverse Laplace transform** of $\overline{f}(s)$ and is denoted by $L^{-1}[\overline{f}(s)]$.

Thus we can say that,

$$L[f(t)] = \overline{f}(s) \Leftrightarrow L^{-1}[\overline{f}(s)] = f(t)$$

Observe the following illustrations.

$$L(1) = \frac{1}{s} \Rightarrow L^{-1}\left(\frac{1}{s}\right) = 1$$

$$L(\cos at) = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

We revert the table of Laplace transforms of standard functions given earlier to present the basic table of inverse Laplace transforms.

	Function	Inverse Transform		Function	Inverse Transform
1	$\frac{1}{s}$	1	5	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
2	$\frac{1}{s-a}$	e^{at}	6	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
3	$\frac{s}{s^2+a^2}$	$\cos at$	7	$\frac{1}{s^{n+1}}$ $(n>-1)$	$\frac{t^n}{\Gamma(n+1)}$
4	$\frac{s}{s^2-a^2}$	$\cosh at$	8	$\frac{1}{s^{n+1}}$ $n=1,2,3..,$	$\frac{t^n}{n!}$

We present a few illustrative examples based on this table of inverse Laplace transforms.

1. $L^{-1}\left(\frac{1}{s-1}\right) = e^t$
2. $L^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$
3. $L^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t$
4. $L^{-1}\left(\frac{s}{s^2-16}\right) = \cosh 4t$
5. $L^{-1}\left(\frac{1}{s^2+5}\right) = \frac{1}{\sqrt{5}}\sin(\sqrt{5}\,t)$
6. $L^{-1}\left(\frac{1}{s^2-36}\right) = \frac{1}{6}\sinh 6t$
7. $L^{-1}\left(\frac{1}{s^4}\right) = \frac{t^3}{3!}$
8. $L^{-1}\left(\frac{1}{s^{3/2}}\right) = \frac{t^{1/2}}{\Gamma(3/2)} = \frac{t^{1/2}}{1/2\cdot\sqrt{\pi}} = 2\sqrt{t/\pi}$

Property : $L^{-1}[c_1\bar{f}(s)+c_2\bar{g}(s)] = c_1 L^{-1}[\bar{f}(s)]+c_2 L^{-1}[\bar{g}(s)]$

WORKED PROBLEMS

Find the inverse Laplace transform of the following functions

1. $\frac{1}{s+2}+\frac{3}{2s+5}-\frac{4}{3s-2}$
2. $\frac{s+2}{s^2+36}+\frac{4s-1}{s^2+25}$
3. $\frac{2s-5}{4s^2+25}+\frac{8-6s}{16s^2+9}$
4. $\frac{2s-5}{8s^2-50}+\frac{4s}{9-s^2}$

5. $\dfrac{(s+2)^3}{s^6}$

6. $\dfrac{3(s^2-1)^2}{2s^5}$

7. $\dfrac{1}{s\sqrt{s}}+\dfrac{3}{s^2\sqrt{s}}-\dfrac{8}{\sqrt{s}}$

8. $\dfrac{3s+5\sqrt{2}}{s^2+8}$

1. $L^{-1}\left[\dfrac{1}{s+2}\right]+\dfrac{3}{2}L^{-1}\left[\dfrac{1}{s+5/2}\right]-\dfrac{4}{3}L^{-1}\left[\dfrac{1}{s-2/3}\right]$

$= e^{-2t}+3/2\cdot e^{-5t/2}-4/3\cdot e^{2t/3}$

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2. $L^{-1}\left[\dfrac{s}{s^2+6^2}\right]+2L^{-1}\left[\dfrac{1}{s^2+6^2}\right]+4L^{-1}\left[\dfrac{s}{s^2+5^2}\right]-L^{-1}\left[\dfrac{1}{s^2+5^2}\right]$

$= \cos 6t+1/3\cdot\sin 6t+4\cos 5t-1/5\cdot\sin 5t$

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3. $2L^{-1}\left[\dfrac{s}{4s^2+25}\right]-5L^{-1}\left[\dfrac{1}{4s^2+25}\right]$

$= \dfrac{1}{2}L^{-1}\left[\dfrac{s}{s^2+(5/2)^2}\right]-\dfrac{5}{4}L^{-1}\left[\dfrac{1}{s^2+(5/2)^2}\right]$

$+\dfrac{1}{2}L^{-1}\left[\dfrac{1}{s^2+(3/4)^2}\right]-\dfrac{3}{8}L^{-1}\left[\dfrac{s}{s^2+(3/4)^2}\right]$

$= \dfrac{1}{2}\cos(5t/2)-\dfrac{5}{4}\cdot\dfrac{2}{5}\sin(5t/2)+\dfrac{1}{2}\cdot\dfrac{4}{3}\sin(3t/4)-\dfrac{3}{8}\cos(3t/4)$

$= 1/2\cdot\cos(5t/2)-1/2\cdot\sin(5t/2)+2/3\cdot\sin(3t/4)-3/8\cdot\cos(3t/4)$

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4. $L^{-1}\left[\dfrac{2s-5}{2(4s^2-25)}\right]-L^{-1}\left[\dfrac{4s}{s^2-9}\right]$

$= \dfrac{1}{2}L^{-1}\left[\dfrac{1}{2s+5}\right]-4L^{-1}\left[\dfrac{s}{s^2-3^2}\right]$

$= 1/4\cdot e^{-5t/2}-4\cosh 3t$

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5. $L^{-1}\left[\dfrac{s^3+6s^2+12s+8}{s^6}\right]$

$= L^{-1}\left(\dfrac{1}{s^3}\right)+6L^{-1}\left(\dfrac{1}{s^4}\right)+12L^{-1}\left(\dfrac{1}{s^5}\right)+8L^{-1}\left(\dfrac{1}{s^6}\right)$

$$= \frac{t^2}{2!} + 6 \cdot \frac{t^3}{3!} + 12 \cdot \frac{t^4}{4!} + 8 \cdot \frac{t^5}{5!}$$

$$= \frac{t^2}{2} + t^3 + \frac{t^4}{2} + \frac{t^5}{15}$$

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6. $\frac{3}{2} L^{-1}\left[\frac{s^4 - 2s^2 + 1}{s^5}\right]$

$$= \frac{3}{2}\left[L^{-1}\left(\frac{1}{s}\right) - 2L^{-1}\left(\frac{1}{s^3}\right) + L^{-1}\left(\frac{1}{s^5}\right)\right]$$

$$= \frac{3}{2}\left[1 - 2 \cdot \frac{t^2}{2!} + \frac{t^4}{4!}\right] = \frac{3}{2}\left[1 - t^2 + \frac{t^4}{24}\right]$$

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7. $L^{-1}\left(\frac{1}{s^{3/2}}\right) + 3L^{-1}\left(\frac{1}{s^{5/2}}\right) - 8\,L^{-1}\left(\frac{1}{s^{1/2}}\right)$

$$= \frac{t^{1/2}}{\Gamma(3/2)} + 3 \cdot \frac{t^{3/2}}{\Gamma(5/2)} - 8\,\frac{t^{-1/2}}{\Gamma(1/2)}$$

$$= \frac{\sqrt{t}}{1/2 \cdot \Gamma(1/2)} + 3\,\frac{t\sqrt{t}}{3/2 \cdot 1/2 \cdot \Gamma(1/2)} - \frac{8}{\sqrt{t}\,\Gamma(1/2)}$$

$$= \frac{2\sqrt{t}}{\sqrt{\pi}} + 4\,\frac{t\sqrt{t}}{\sqrt{\pi}} - \frac{8}{\sqrt{t}\sqrt{\pi}}$$

$$= \frac{2}{\sqrt{\pi}}\left[\sqrt{t} + 2t\sqrt{t} - \frac{4}{\sqrt{t}}\right]$$

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8. $3L^{-1}\left[\frac{s}{s^2 + (\sqrt{8})^2}\right] + 5\sqrt{2}\,L^{-1}\left[\frac{1}{s^2 + (\sqrt{8})^2}\right]$

$$= 3\cos(2\sqrt{2}\,t) + 5/2 \cdot \sin(2\sqrt{2}\,t)$$

8.21 Computation of the inverse transform of $e^{-as}\,\overline{f}(s)$

We have proved that $L[f(t-a)\,u(t-a)] = e^{-as}\,\overline{f}(s)$

$\therefore \quad L^{-1}[e^{-as}\,\overline{f}(s)] = f(t-a)\,u(t-a)$

Working procedure for problems

- In the given function we should observe the presence of e^{-as} first and identify the remaining part of the function to be called as $\overline{f}(s)$.

➲ Taking the inverse of $\bar{f}(s)$ we obtain $f(t)$.

➲ The required inverse of $e^{-as}\bar{f}(s)$ is obtained by replacing t by $(t-a)$ in $f(t)$ to be multiplied by the unit step function $u(t-a)$

WORKED PROBLEMS

Find the inverse Laplace transforms of the following

9. $\dfrac{1+e^{-3s}}{s^2}$

10. $\dfrac{3}{s^2}+\dfrac{2e^{-s}}{s^3}-\dfrac{3e^{-2s}}{s}$

11. $\dfrac{\cosh 2s}{e^{3s}s^2}$

12. $\dfrac{(1-e^{-s})(2-e^{-2s})}{s^3}$

13. $\dfrac{e^{-\pi s}}{s^2+1}+\dfrac{s\,e^{-2\pi s}}{s^2+4}$

14. $\dfrac{s\,e^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$

9. $L^{-1}\left(\dfrac{1}{s^2}\right)+L^{-1}\left(\dfrac{e^{-3s}}{s^2}\right)$. **We have** $L^{-1}\left(1/s^2\right)=t$

Thus $L^{-1}\left[\dfrac{1+e^{-3s}}{s^2}\right]=t+(t-3)\,u(t-3)$

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10. $3L^{-1}\left(\dfrac{1}{s^2}\right)+2L^{-1}\left(\dfrac{e^{-s}}{s^3}\right)-3L^{-1}\left(\dfrac{e^{-2s}}{s}\right)$ ... (1)

We have $L^{-1}\left(\dfrac{1}{s^2}\right)=t,\ L^{-1}\left(\dfrac{1}{s^3}\right)=\dfrac{t^2}{2},\ L^{-1}\left(\dfrac{1}{s}\right)=1$

Hence (1) becomes,

$$3\cdot t+2\frac{(t-1)^2}{2}u(t-1)-3\cdot 1\cdot u(t-2)$$

Thus $L^{-1}\left[\dfrac{3}{s^2}+\dfrac{2e^{-s}}{s^3}-\dfrac{3e^{-2s}}{s}\right]=3t+(t-1)^2u(t-1)-3u(t-2)$

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11. $\dfrac{\cosh 2s}{e^{3s}\cdot s^2} = \dfrac{e^{-3s}}{s^2}\dfrac{(e^{2s}+e^{-2s})}{2} = \dfrac{1}{2}\left[\dfrac{e^{-s}}{s^2}+\dfrac{e^{-5s}}{s^2}\right]$

Now, $L^{-1}\left[\dfrac{\cosh 2s}{e^{3s}s^2}\right] = \dfrac{1}{2}\left\{L^{-1}\left(\dfrac{e^{-s}}{s^2}\right)+L^{-1}\left(\dfrac{e^{-5s}}{s^2}\right)\right\}$. But $L^{-1}\left(1/s^2\right) = t$

Thus $L^{-1}\left[\dfrac{\cosh 2s}{e^{3s}\cdot s^2}\right] = \dfrac{1}{2}\{(t-1)u(t-1)+(t-5)u(t-5)\}$

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12. $\dfrac{(1-e^{-s})(2-e^{-2s})}{s^3} = \dfrac{2-2e^{-s}-e^{-2s}+e^{-3s}}{s^3}$

Now, $L^{-1}\left[\dfrac{(1-e^{-s})(2-e^{-2s})}{s^3}\right]$

$$= 2L^{-1}\left(\frac{1}{s^3}\right)-2L^{-1}\left(\frac{e^{-s}}{s^3}\right)-L^{-1}\left(\frac{e^{-2s}}{s^3}\right)+L^{-1}\left(\frac{e^{-3s}}{s^3}\right)$$

But $L^{-1}\left(1/s^3\right) = t^2/2$

Thus $L^{-1}\left[\dfrac{(1-e^{-s})(2-e^{-2s})}{s^3}\right]$

$$= t^2-(t-1)^2u(t-1)-\frac{(t-2)^2u(t-2)}{2}+\frac{(t-3)^2u(t-3)}{2}$$

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13. $L^{-1}\left(\dfrac{e^{-\pi s}}{s^2+1}\right)+L^{-1}\left(e^{-2\pi s}\cdot\dfrac{s}{s^2+4}\right)$ ... (1)

We have $L^{-1}\left(\dfrac{1}{s^2+1}\right) = \sin t,\ L^{-1}\left(\dfrac{s}{s^2+4}\right) = \cos 2t$

Hence (1) becomes,

$\sin(t-\pi)u(t-\pi)+\cos 2(t-2\pi)u(t-2\pi)$

Thus $L^{-1}\left[\dfrac{e^{-\pi s}}{s^2+1}+\dfrac{s\,e^{-2\pi s}}{s^2+1}\right] = -\sin t\ u(t-\pi)+\cos 2t\ u(t-2\pi)$

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14. $L^{-1}\left(e^{-s/2}\cdot\dfrac{s}{s^2+\pi^2}\right)+L^{-1}\left(e^{-s}\cdot\dfrac{\pi}{s^2+\pi^2}\right)$ ... (1)

We have $L^{-1}\left(\dfrac{s}{s^2+\pi^2}\right)=\cos\pi t,\ L^{-1}\left(\dfrac{\pi}{s^2+\pi^2}\right)=\sin\pi t$

Hence (1) becomes

$$\cos\pi(t-1/2)\,u(t-1/2)+\sin\pi(t-1)\,u(t-1)$$

$$=\sin\pi t\ u(t-1/2)-\sin\pi t\ u(t-1)$$

Thus $L^{-1}\left[\dfrac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right]=\mathbf{\sin\pi t[u(t-1/2)-u(t-1)]}$

8.22 Inverse transform by completing the square

We have the property that if $L[f(t)]=\bar{f}(s)$ then $L[e^{at}f(t)]=\bar{f}(s-a)$.

This implies that

$$L^{-1}[\bar{f}(s)]=f(t) \quad ...(1)$$

and $$L^{-1}[\bar{f}(s-a)]=e^{at}f(t) \quad ...(2)$$

(2) as a consequence of (1) becomes

$$L^{-1}[\bar{f}(s-a)]=e^{at}L^{-1}[\bar{f}(s)] \quad ...(3)$$

Working procedure for problems

- Given $\bar{f}(s)=\dfrac{\phi(s)}{(ps^2+qs+r)}$,

we first express (ps^2+qs+r) in the form $(s-a)^2\pm b^2$ and later express $\phi(s)$ in terms of $(s-a)$ so that the given function of s reduces to a function of $(s-a)$.

- We then use (3) to obtain the result.
- However if $\bar{f}(s)=\phi(s)/\psi(s-a)$ we only need to express $\phi(s)$ in terms of $(s-a)$ to compute the inverse transform.

WORKED PROBLEMS

Find the inverse Laplace transform of the following functions

15. $\dfrac{s+5}{s^2-6s+13}$

16. $\dfrac{s^2}{(s+1)^3}$

17. $\dfrac{(s+2)e^{-s}}{(s+1)^4}$

18. $\dfrac{2s-1}{s^2+4s+29}$

19. $\dfrac{s+1}{s^2+6s+9}$

20. $\dfrac{e^{-4s}}{(s-4)^2}$

21. $\dfrac{s+1}{s^2+s+1}$

22. $\dfrac{2s+1}{s^2+3s+1}$

23. $\dfrac{7s+4}{4s^2+4s+9}$

24. $\dfrac{1}{(s+4)^{5/2}}+\dfrac{1}{\sqrt{2s+3}}$

15. $L^{-1}\left[\dfrac{s+5}{s^2-6s+13}\right]=L^{-1}\left[\dfrac{s+5}{(s-3)^2+4}\right]$

ie., $=L^{-1}\left[\dfrac{\overline{s-3}+3+5}{(s-3)^2+2^2}\right]=L^{-1}\left[\dfrac{(s-3)+8}{(s-3)^2+2^2}\right]$

Here $a=3$ and $(s-3)$ changes to s

ie., $=e^{3t}L^{-1}\left[\dfrac{s+8}{s^2+2^2}\right]$

$=e^{3t}\left\{L^{-1}\left(\dfrac{s}{s^2+2^2}\right)+8L^{-1}\left(\dfrac{1}{s^2+2^2}\right)\right\}$

Thus $\boldsymbol{L^{-1}\left[\dfrac{s+5}{s^2-6s+13}\right]=e^{3t}(\cos 2t+4\sin 2t)}$

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16. Here we need to express s^2 in terms of $(s+1)$

$s^2=(s+1)^2-2s-1=(s+1)^2-2(s+1)+2-1$

ie., $s^2=(s+1)^2-2(s+1)+1.$

$\therefore\quad L^{-1}\left[\dfrac{s^2}{(s+1)^3}\right]=L^{-1}\left[\dfrac{(s+1)^2-2(s+1)+1}{(s+1)^3}\right]$

Here $a = -1$ and $(s+1)$ changes to s. Hence R.H.S becomes

$$e^{-t} L^{-1}\left[\frac{s^2-2s+1}{s^3}\right] = e^{-t}\left\{L^{-1}\left(\frac{1}{s}\right) - 2L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{1}{s^3}\right)\right\}$$

Thus $$L^{-1}\left[\frac{s^2}{(s+1)^3}\right] = e^{-t}\left(1-2t+\frac{t^2}{2}\right)$$

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17. Let $\bar{f}(s) = \dfrac{s+2}{(s+1)^4}$

We shall first find $L^{-1}[\bar{f}(s)] = f(t)$

$$L^{-1}\left[\frac{s+2}{(s+1)^4}\right] = L^{-1}\left[\frac{(s+1)+1}{(s+1)^4}\right] = e^{-t} L^{-1}\left[\frac{s+1}{s^4}\right]$$

ie., $$L^{-1}\left[\frac{s+2}{(s+1)^4}\right] = e^{-t}\left\{L^{-1}\left(\frac{1}{s^3}\right) + L^{-1}\left(\frac{1}{s^4}\right)\right\}.$$

Using $L^{-1}\left(\dfrac{1}{s^{n+1}}\right) = \dfrac{t^n}{n!}$ we get

$$f(t) = e^{-t}\left(\frac{t^2}{2!}+\frac{t^3}{3!}\right) = e^{-t}\left(\frac{t^2}{2}+\frac{t^3}{6}\right)$$

Next we have $L^{-1}[e^{-s}\bar{f}(s)] = f(t-1)\,u(t-1)$

Thus $$L^{-1}\left[e^{-s}\frac{s+2}{(s+1)^4}\right] = e^{-(t-1)}\left\{\frac{(t-1)^2}{2}+\frac{(t-1)^3}{6}\right\}u(t-1)$$

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18. $$L^{-1}\left\{\frac{2s-1}{s^2+4s+29}\right\} = L^{-1}\left\{\frac{2s-1}{(s+2)^2+25}\right\} = L^{-1}\left\{\frac{2(s+2)-5}{(s+2)^2+25}\right\}$$

ie., $$= e^{-2t} L^{-1}\left\{\frac{2s-5}{s^2+5^2}\right\} = e^{-2t}\left\{2L^{-1}\left(\frac{s}{s^2+5^2}\right) - L^{-1}\left(\frac{5}{s^2+5^2}\right)\right\}$$

Thus $$L^{-1}\left[\frac{2s-1}{s^2+4s+29}\right] = e^{-2t}(2\cos 5t - \sin 5t)$$

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19. $L^{-1}\left[\dfrac{s+1}{s^2+6s+9}\right] = L^{-1}\left[\dfrac{(s+3)-2}{(s+3)^2}\right] = e^{-3t}L^{-1}\left[\dfrac{s-2}{s^2}\right]$

ie., $= e^{-3t}\left\{L^{-1}\left(\dfrac{1}{s}\right) - 2L^{-1}\left(\dfrac{1}{s^2}\right)\right\}$

Thus $\mathbf{L^{-1}\left[\dfrac{s+1}{s^2+6s+9}\right] = e^{-3t}(1-2t)}$

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20. Let $\bar{f}(s) = \dfrac{1}{(s-4)^2}$

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{(s-4)^2}\right] = e^{4t}L^{-1}\left(\frac{1}{s^2}\right) = e^{4t}\cdot t = f(t)$$

But $L^{-1}[e^{-4s}\bar{f}(s)] = f(t-4)\,u(t-4)$

Thus $\mathbf{L^{-1}\left[\dfrac{e^{-4s}}{(s-4)^2}\right] = \left\{e^{4(t-4)}(t-4)\right\}u(t-4)}$

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21. $s^2+s+1 = (s+1/2)^2 - 1/4 + 1 = (s+1/2)^2 + (\sqrt{3}/2)^2$

$$L^{-1}\left[\frac{s+1}{s^2+s+1}\right] = L^{-1}\left[\frac{(s+1/2)+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right] = e^{-t/2}L^{-1}\left[\frac{s+1/2}{s^2+(\sqrt{3}/2)^2}\right]$$

ie., $= e^{-t/2}\left\{L^{-1}\left[\dfrac{s}{s^2+(\sqrt{3}/2)^2}\right] + \dfrac{1}{2}L^{-1}\left[\dfrac{1}{s^2+(\sqrt{3}/2)^2}\right]\right\}$

Thus $\mathbf{L^{-1}\left[\dfrac{s+1}{s^2+s+1}\right] = e^{-t/2}\left\{\cos(\sqrt{3}\,t/2) + 1/\sqrt{3}\cdot\sin(\sqrt{3}\,t/2)\right\}}$

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22. $s^2+3s+1 = (s+3/2)^2 - 9/4 + 1 = (s+3/2)^2 - (\sqrt{5}/2)^2$

$$L^{-1}\left[\frac{2s+1}{s^2+3s+1}\right] = L^{-1}\left[\frac{2s+1}{(s+3/2)^2-(\sqrt{5}/2)^2}\right]$$

$$= L^{-1}\left[\frac{2(s+3/2)-2}{(s+3/2)^2-(\sqrt{5}/2)^2}\right]$$

$$= e^{-3t/2}\left\{2L^{-1}\left[\frac{s}{s^2-(\sqrt{5}/2)^2}\right] - 2L^{-1}\left[\frac{1}{s^2-(\sqrt{5}/2)^2}\right]\right\}$$

$\mathbf{L^{-1}\left[\dfrac{2s+1}{s^2+3s+1}\right] = e^{-3t/2}\left\{2\cosh(\sqrt{5}\,t/2) - 4/\sqrt{5}\cdot\sinh(\sqrt{5}\,t/2)\right\}}$

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23. Consider $4s^2 + 4s + 9$

$$= 4\,(s^2 + s + 9/4) = 4\left\{(s + 1/2)^2 + 2\right\}$$

Hence $7s + 4 = 7\,(s + 1/2) + 1/2$

Now $L^{-1}\left[\dfrac{7s+4}{4s^2+4s+9}\right] = \dfrac{1}{4}L^{-1}\left[\dfrac{7\,(s+1/2)+1/2}{(s+1/2)^2+2}\right]$

ie., $= \dfrac{e^{-t/2}}{4}\left\{7L^{-1}\left[\dfrac{s}{s^2+(\sqrt{2}\,)^2}\right] + \dfrac{1}{2}L^{-1}\left[\dfrac{1}{s^2+(\sqrt{2}\,)^2}\right]\right\}$

Thus $L^{-1}\left[\dfrac{7s+4}{4s^2+4s+9}\right] = \dfrac{e^{-t/2}}{4}\left\{7\cos(\sqrt{2}\,t) + \dfrac{1}{2\sqrt{2}}\sin(\sqrt{2}\,t)\right\}$

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24. $L^{-1}\left[\dfrac{1}{(s+4)^{5/2}}\right] + L^{-1}\left[\dfrac{1}{\sqrt{2s+3}}\right]$

$$= e^{-4t}L^{-1}\left[\frac{1}{s^{5/2}}\right] + \frac{1}{\sqrt{2}}L^{-1}\left[\frac{1}{\sqrt{s+3/2}}\right]$$

$$= e^{-4t}\left[\frac{t^{3/2}}{\Gamma(5/2)}\right] + \frac{e^{-3t/2}}{\sqrt{2}}L^{-1}\left(\frac{1}{\sqrt{s}}\right).$$

But $\Gamma(5/2) = \dfrac{3}{2}\cdot\dfrac{1}{2}\cdot\sqrt{\pi} = \dfrac{3\sqrt{\pi}}{4}$

Thus we have $L^{-1}\left[\dfrac{1}{(s+4)^{5/2}}\right] + L^{-1}\left[\dfrac{1}{\sqrt{2s+3}}\right]$

$$= \frac{4}{3\sqrt{\pi}}e^{-4t}\,t^{3/2} + \frac{e^{-3t/2}}{\sqrt{2}}\,\frac{t^{-1/2}}{\Gamma(1/2)} \quad \text{But } \Gamma(1/2) = \sqrt{\pi}$$

$$= \frac{1}{\sqrt{\pi}}\left\{\frac{4}{3}e^{-4t}\,t^{3/2} + \frac{e^{-3t/2}}{\sqrt{2t}}\right\}$$

Find the inverse Laplace transform of the following functions.

25. $\dfrac{s}{s^4 + 4a^4}$

26. $\dfrac{s^2}{s^4 + 4a^4}$

27. $\dfrac{2\,(s^2 + 2a^2)\,e^{-2s}}{s^4 + 4a^4}$

28. $\dfrac{s}{s^4 + s^2 + 1}$

Note : *In these problems we factorize the denominator and express the numerator interms of the factors of the denominator by simple adjustment.*

25. $s^4 + 4a^4 = (s^2 + 2a^2)^2 - 4a^2 s^2$

ie., $s^4 + 4a^4 = (s^2 + 2a^2 + 2as)(s^2 + 2a^2 - 2as)$ . . . (1)

Also $4as = (s^2 + 2a^2 + 2as) - (s^2 + 2a^2 - 2as)$

Now $$\frac{s}{s^4 + 4a^4} = \frac{1}{4a}\left\{\frac{(s^2 + 2a^2 + 2as) - (s^2 + 2a^2 - 2as)}{(s^2 + 2a^2 + 2as)(s^2 + 2a^2 - 2as)}\right\}$$

ie., $$\frac{s}{s^4 + 4a^4} = \frac{1}{4a}\left\{\frac{1}{s^2 + 2a^2 - 2as} - \frac{1}{s^2 + 2a^2 + 2as}\right\}$$

$$L^{-1}\left[\frac{s}{s^4 + 4a^4}\right] = \frac{1}{4a}\left\{L^{-1}\left[\frac{1}{(s-a)^2 + a^2}\right] - L^{-1}\left[\frac{1}{(s+a)^2 + a^2}\right]\right\}$$

$$= \frac{1}{4a}\left\{e^{at} L^{-1}\left[\frac{1}{s^2 + a^2}\right] - e^{-at} L^{-1}\left[\frac{1}{s^2 + a^2}\right]\right\}$$

$$= \frac{1}{4a}\left\{e^{at}\frac{\sin at}{a} - e^{-at}\frac{\sin at}{a}\right\}$$

Thus $$L^{-1}\left[\frac{s}{s^4 + 4a^4}\right] = \frac{\sin at}{2a^2}\left\{\frac{e^{at} - e^{-at}}{2}\right\} = \frac{\sin at \sinh at}{2a^2}$$

---

26. As in the previous problem, we have,

$$\frac{s}{s^4 + 4a^4} = \frac{1}{4a}\left\{\frac{1}{(s-a)^2 + a^2} - \frac{1}{(s+a)^2 + a^2}\right\}.$$

Multiplying by s we have

$$\frac{s^2}{s^4 + 4a^4} = \frac{1}{4a}\left\{\frac{s}{(s-a)^2 + a^2} - \frac{s}{(s+a)^2 + a^2}\right\}$$

or $$\frac{s^2}{s^4 + 4a^4} = \frac{1}{4a}\left\{\frac{(s-a) + a}{(s-a)^2 + a^2} - \frac{(s+a) - a}{(s+a)^2 + a^2}\right\}$$

$\therefore$ $$L^{-1}\left[\frac{s^2}{s^4 + 4a^4}\right] = \frac{1}{4a}\left\{e^{at} L^{-1}\left[\frac{s+a}{s^2 + a^2}\right] - e^{-at} L^{-1}\left[\frac{s-a}{s^2 + a^2}\right]\right\}$$

ie., $$= \frac{1}{4a}\left\{e^{at}(\cos at + \sin at) - e^{-at}(\cos at - \sin at)\right\}$$

$$= \frac{1}{2a}\left\{\frac{e^{at} - e^{-at}}{2}\cos at + \frac{e^{at} + e^{-at}}{2}\sin at\right\}$$

Thus $$L^{-1}\left[\frac{s^2}{s^4 + 4a^4}\right] = \frac{1}{2a}(\sinh at \cos at + \cosh at \sin at)$$

---

27. We have $s^4 + 4a^4 = (s^2 + 2a^2 + 2as)(s^2 + 2a^2 - 2as)$

Also $2s^2 + 4a^2 = (s^2 + 2a^2 + 2as) + (s^2 + 2a^2 - 2as)$

$$\therefore \quad L^{-1}\left[\frac{2(s^2 + 2a^2)}{s^4 + 4a^4}\right] = L^{-1}\left[\frac{(s^2 + 2a^2 + 2as) + (s^2 + 2a^2 - 2as)}{(s^2 + 2a^2 + 2as)(s^2 + 2a^2 - 2as)}\right]$$

$$= L^{-1}\left[\frac{1}{s^2 + 2a^2 - 2as} + \frac{1}{s^2 + 2a^2 + 2as}\right]$$

$$= L^{-1}\left[\frac{1}{(s-a)^2 + a^2}\right] + L^{-1}\left[\frac{1}{(s+a)^2 + a^2}\right]$$

$$= e^{at} L^{-1}\left[\frac{1}{s^2 + a^2}\right] + e^{-at} L^{-1}\left[\frac{1}{s^2 + a^2}\right]$$

$$= e^{at} \cdot \frac{\sin at}{a} + e^{-at} \cdot \frac{\sin at}{a} = \frac{\sin at}{a}(e^{at} + e^{-at}) = \frac{2}{a}\sin at \cosh at$$

Thus $L^{-1}\left[\frac{2(s^2 + 2a^2)}{s^4 + 4a^4} e^{-2s}\right] = \frac{2}{a}\left\{\sin a(t-2)\cosh a(t-2)\right\} u(t-2)$

---

28. $s^4 + s^2 + 1 = (s^2 + 1)^2 - s^2 = (s^2 + 1 - s)(s^2 + 1 + s)$

Also $2s = (s^2 + 1 + s) - (s^2 + 1 - s)$

$$\therefore \quad \frac{s}{s^4 + s^2 + 1} = \frac{1}{2}\left[\frac{(s^2 + 1 + s) - (s^2 + 1 - s)}{(s^2 + 1 + s)(s^2 + 1 - s)}\right]$$

ie., $$\frac{s}{s^4 + s^2 + 1} = \frac{1}{2}\left[\frac{1}{s^2 - s + 1} - \frac{1}{s^2 + s + 1}\right]$$

Now $$L^{-1}\left[\frac{s}{s^4 + s^2 + 1}\right] = \frac{1}{2}\left\{L^{-1}\left[\frac{1}{s^2 - s + 1}\right] - L^{-1}\left[\frac{1}{s^2 + s + 1}\right]\right\}$$

ie., $$= \frac{1}{2}\left\{L^{-1}\left[\frac{1}{(s - 1/2)^2 + (\sqrt{3}/2)^2}\right] - L^{-1}\left[\frac{1}{(s + 1/2)^2 + (\sqrt{3}/2)^2}\right]\right\}$$

$$= \frac{1}{2}\left\{e^{t/2} L^{-1}\left[\frac{1}{s^2 + (\sqrt{3}/2)^2}\right] - e^{-t/2} L^{-1}\left[\frac{1}{s^2 + (\sqrt{3}/2)^2}\right]\right\}$$

$$= \frac{1}{2}\left\{e^{t/2} \cdot \frac{2}{\sqrt{3}} \sin(\sqrt{3}\, t/2) - e^{-t/2} \cdot \frac{2}{\sqrt{3}} \sin(\sqrt{3}\, t/2)\right\}$$

$$= \frac{2}{\sqrt{3}} \sin(\sqrt{3}\, t/2)\left\{\frac{e^{t/2} - e^{-t/2}}{2}\right\}$$

Thus $L^{-1}\left[\frac{s}{s^4 + s^2 + 1}\right] = \frac{2}{\sqrt{3}} \sin(\sqrt{3}\, t/2) \cdot \sinh(t/2)$

8.23 Inverse transform by the method of partial fractions

We know that, the method of partial fractions is a technique of converting an algebraic fraction $\phi(s)/\psi(s)$ [*where degree of* $\phi(s)$ *is less than that of* $\psi(s)$] into a sum. Depending on the nature of terms in $\psi(s)$ we have to split into a sum of various terms with constants $A, B, C, D, \cdots$ which can be determined. Later the inverse is found term by term.

Find the inverse Laplace transform of the following functions.

29. $\dfrac{1}{s(s+1)(s+2)(s+3)}$

30. $\dfrac{2s^2+5s-4}{s^3+s^2-2s}$

31. $\dfrac{3s+2}{s^2-s-2}$

32. $\dfrac{s^2}{(s^2+1)(s^2+4)}$

33. $\dfrac{4s+5}{(s+1)^2(s+2)}$

34. $\dfrac{s+2}{s^2(s+3)}$

35. $\dfrac{(3s+1)e^{-3s}}{(s-1)(s^2+1)}$

36. $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

37. $\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$

29. Let $\dfrac{1}{s(s+1)(s+2)(s+3)} = \dfrac{A}{s}+\dfrac{B}{s+1}+\dfrac{C}{s+2}+\dfrac{D}{s+3}$

Multiplying by $s(s+1)(s+2)(s+3)$ we get,

$$1 = A(s+1)(s+2)(s+3)+Bs(s+2)(s+3) + Cs(s+1)(s+3)+Ds(s+1)(s+2)$$

Put $s=0$: $1=A(6)$ $\therefore A=1/6$

Put $s=-1$: $1=B(-2)$ $\therefore B=-1/2$

Put $s=-2$: $1=C(2)$ $\therefore C=1/2$

Put $s=-3$: $1=D(-6)$ $\therefore D=-1/6$

Now, $L^{-1}\left[\dfrac{1}{s(s+1)(s+2)(s+3)}\right]$

$$= \frac{1}{6}L^{-1}\left[\frac{1}{s}\right]-\frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right]-\frac{1}{6}L^{-1}\left[\frac{1}{s+3}\right]$$

Thus $$L^{-1}\left[\frac{1}{s(s+1)(s+2)(s+3)}\right] = \frac{1}{6}-\frac{1}{2}e^{-t}+\frac{1}{2}e^{-2t}-\frac{1}{6}e^{-3t}$$

---

30. $s^3+s^2-2s = s(s^2+s-2) = s(s-1)(s+2)$

Let $$\frac{2s^2+5s-4}{s(s-1)(s+2)} = \frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+2}$$

Multiplying by $s(s-1)(s+2)$ we get,

$$2s^2+5s-4 = A(s-1)(s+2)+Bs(s+2)+Cs(s-1)$$

Put $s=0$: $-4 = A(-2)$ $\therefore$ $A=2$

Put $s=1$: $3 = B(3)$ $\therefore$ $B=1$

Put $s=-2$: $-6 = C(6)$ $\therefore$ $C=-1$

Now, $$L^{-1}\left[\frac{2s^2+5s-4}{s(s-1)(s+2)}\right]$$

$$= 2L^{-1}\left[\frac{1}{s}\right]+L^{-1}\left[\frac{1}{s-1}\right]-L^{-1}\left[\frac{1}{s+2}\right]$$

Thus $$L^{-1}\left[\frac{2s^2+5s-4}{s(s-1)(s+2)}\right] = 2+e^t-e^{-2t}$$

---

31. Note : *The problem can be done by completing the square. Since the quadratic is factorizable the method of partial fractions is preferred.*

Let $$\frac{3s+2}{(s-2)(s+1)} = \frac{A}{s-2}+\frac{B}{s+1}$$

or $$3s+2 = A(s+1)+B(s-2)$$

Put $s=2$: $8 = A(3)$ $\therefore$ $A = 8/3$

Put $s=-1$: $-1 = B(-3)$ $\therefore$ $B = 1/3$

$$L^{-1}\left[\frac{3s+2}{(s-2)(s+1)}\right] = \frac{8}{3}L^{-1}\left[\frac{1}{s-2}\right]+\frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right]$$

Thus $$L^{-1}\left[\frac{3s+2}{(s-2)(s+1)}\right] = \frac{1}{3}\left(8e^{2t}+e^{-t}\right)$$

---

32. We have $\dfrac{s^2}{(s^2+1)(s^2+4)}$ and let $s^2 = t$ for convenience.

We now have $\dfrac{t}{(t+1)(t+4)} = \dfrac{A}{t+1} + \dfrac{B}{t+4}$ *(say)*

or $\quad t = A(t+4) + B(t+1)$

Put $t = -1$: $-1 = A(3)$ $\therefore$ $A = -1/3$

Put $t = -4$: $-4 = B(-3)$ $\therefore$ $B = 4/3$

Hence $\dfrac{t}{(t+1)(t+4)} = \dfrac{-1}{3} \cdot \dfrac{1}{t+1} + \dfrac{4}{3} \cdot \dfrac{1}{t+4}$

Substituting $t = s^2$ and taking inverse we have,

$$L^{-1}\left[\frac{s^2}{(s^2+1)(s^2+4)}\right]$$

$$= \frac{-1}{3} L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{4}{3} L^{-1}\left[\frac{1}{s^2+4}\right]$$

Thus, $\mathbf{L^{-1}\left[\dfrac{s^2}{(s^2+1)(s^2+4)}\right] = \dfrac{1}{3}(2\sin 2t - \sin t)}$

---

33. Let $\dfrac{4s+5}{(s+1)^2(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{(s+1)^2} + \dfrac{C}{s+2}$

Multiplying by $(s+1)^2(s+2)$ we get,

$$4s+5 = A(s+1)(s+2) + B(s+2) + C(s+1)^2 \quad \ldots (1)$$

Put $s = -1$: $1 = B(1)$ $\therefore$ $B = 1$

Put $s = -2$: $-3 = C(1)$ $\therefore$ $C = -3$

Equating the coefficient of s^2 on both sides of (1) we get,

$$0 = A + C \quad \therefore \quad A = 3$$

Now, $L^{-1}\left[\dfrac{4s+5}{(s+1)^2(s+2)}\right]$

$$= 3L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2}\right] - 3L^{-1}\left[\frac{1}{s+2}\right]$$

$$= 3e^{-t} + e^{-t} L^{-1}[1/s^2] - 3e^{-2t}$$

Thus, $\mathbf{L^{-1}\left[\dfrac{4s+5}{(s+1)^2(s+2)}\right] = 3e^{-t} + e^{-t} \cdot t - 3e^{-2t}}$

---

34. Let $\dfrac{s+2}{s^2(s+3)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s+3}$

Multiplying by $s^2(s+3)$ we get,

$$s+2 = As(s+3) + B(s+3) + Cs^2 \qquad \ldots (1)$$

Put $s = 0$: $2 = B(3)$ $\therefore$ $B = 2/3$

Put $s = -3$: $-1 = C(9)$ $\therefore$ $C = -1/9$

Equating the coefficient of s^2 on both sides of (1) we get,

$$0 = A + C \quad \therefore \quad A = -C = 1/9$$

Now $L^{-1}\left[\dfrac{s+2}{s^2(s+3)}\right] = \dfrac{1}{9}L^{-1}\left[\dfrac{1}{s}\right] + \dfrac{2}{3}L^{-1}\left[\dfrac{1}{s^2}\right] - \dfrac{1}{9}L^{-1}\left[\dfrac{1}{s+3}\right]$

Thus, $\boldsymbol{L^{-1}\left[\dfrac{s+2}{s^2(s+3)}\right] = \dfrac{1}{9}(1+6t-e^{-3t})}$

---

35. Let $\dfrac{3s+1}{(s-1)(s^2+1)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+1}$

or $3s+1 = A(s^2+1) + (Bs+C)(s-1) \qquad \ldots (1)$

Put $s = 1$: $4 = A(2)$ $\therefore$ $A = 2$

Put $s = 0$: $1 = A - C$ $\therefore$ $C = 1$

Equating the coefficient of s^2 on both sides of (1) we get,

$$0 = A + B \quad \therefore \quad B = -2.$$

$$L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right] = 2L^{-1}\left[\frac{1}{s-1}\right] - 2L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 2e^t - 2\cos t + \sin t$$

Thus, $\boldsymbol{L^{-1}\left[\dfrac{(3s+1)e^{-3s}}{(s-1)(s^2+1)}\right] = [2e^{t-3} - 2\cos(t-3) + \sin(t-3)]\,u(t-3)}$

---

36. Let $\dfrac{5s+3}{(s-1)(s^2+2s+5)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+2s+5}$

or $5s+3 = A(s^2+2s+5) + (Bs+C)(s-1) \qquad \ldots (1)$

Put $s = 1$: $8 = A(8)$ $\therefore$ $A = 1$

Put $s = 0$: $3 = 5A - C$ $\therefore$ $C = 2$

3

Equating the coefficient of s^2 on both sides of (1) we get

$$0 = A + B \quad \therefore \quad B = -1$$

Now we have,

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{-s+2}{(s^2+2s+5)} = \frac{1}{s-1} + \frac{2-s}{(s+1)^2+4}$$

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{3-(s+1)}{(s+1)^2+4}$$

$$L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{3-(s+1)}{(s+1)^2+4)}\right]$$

$$= e^t + e^{-t} L^{-1}\left[\frac{3-s}{s^2+4}\right]$$

$$= e^t + e^{-t}\left\{3 L^{-1}\left[\frac{1}{s^2+4}\right] - L^{-1}\left[\frac{s}{s^2+4}\right]\right\}$$

Thus $\quad L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = e^t + e^{-t}\left[\frac{3}{2}\cdot \sin 2t - \cos 2t\right]$

---

37. Consider $\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$

Let $\quad t = s^2 + 2s$ for convenience

$\therefore \quad \dfrac{t+3}{(t+2)(t+5)} = \dfrac{A}{t+2} + \dfrac{B}{t+5}$ (*say*) where A and B are constants.

or $\quad t+3 = A(t+5) + B(t+2)$

Put $t = -2 \;:\; 1 = A(3) \quad \therefore \; A = 1/3$

Put $t = -5 \;:\; -2 = B(-3) \;\therefore\; B = 2/3$

Hence $\dfrac{t+3}{(t+2)(t+5)} = \dfrac{1}{3}\cdot\dfrac{1}{t+2} + \dfrac{2}{3}\cdot\dfrac{1}{t+5}$ where $t = s^2 + 2s$

Now, $L^{-1}\left[\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right]$

$$= \frac{1}{3}L^{-1}\left[\frac{1}{s^2+2s+2}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s^2+2s+5}\right]$$

$$= \frac{1}{3}L^{-1}\left[\frac{1}{(s+1)^2+1}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$= \frac{1}{3}e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{2}{3}e^{-t}L^{-1}\left[\frac{1}{s^2+4}\right]$$

Thus, $L^{-1}\left[\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right] = \dfrac{1}{3}e^{-t}\sin t + \dfrac{1}{3}e^{-t}\sin 2t = \dfrac{e^{-t}}{3}(\sin t + \sin 2t)$

8.24 Inverse transform of logarithmic functions and inverse functions

Given $\overline{f}(s)$ we need to find $L^{-1}[\overline{f}(s)] = f(t)$

We have the property : $L[tf(t)] = -\overline{f}'(s)$

Equivalently, $L^{-1}[-\overline{f}'(s)] = tf(t)$ . . . (1)

Working procedure for problems

- In the case of logarithmic functions we apply the properties of logarithms and then differentiate w.r.t s to obtain $\overline{f}'(s)$.
- We then multiply by – 1 and take inverse on both sides.
- L.H.S becomes $tf(t)$ by (1) and inverses are also found for the terms in R.H.S with the result we obtain the required $f(t)$.
- If logarithmic function persists in $\overline{f}'(s)$ we differentiate again w.r.t s to obtain $f''(s)$ and use the property that

$$L^{-1}[\overline{f}''(s)] = t^2 f(t) \text{ since } L[t^2 f(t)] = \overline{f}''(s)$$

- In the cases of inverse functions we simply differentiale the given $\overline{f}(s)$ and use the result (1) to obtain $f(t)$.

WORKED PROBLEMS

Find the inverse Laplace transform of the following functions

38. $\log\left(\dfrac{s+a}{s+b}\right)$

39. $\log\left(1+\dfrac{a^2}{s^2}\right)$

40. $\cot^{-1}(s/a)$

41. $\log\sqrt{(s^2+1)/(s^2+4)}$

42. $\log\left[\dfrac{s^2+4}{s(s+4)(s-4)}\right]$

43. $\tan^{-1}(2/s^2)$

44. $s\log\left(\dfrac{s+4}{s-4}\right)$

45. $\cot^{-1}\left(\dfrac{s+a}{b}\right)$

38. Let $\bar{f}(s) = \log\left(\dfrac{s+a}{s+b}\right) = \log(s+a) - \log(s+b)$

$\therefore \quad -\bar{f}'(s) = -\left\{\dfrac{1}{s+a} - \dfrac{1}{s+b}\right\} = \dfrac{1}{s+b} - \dfrac{1}{s+a}$

Now $\quad L^{-1}[-\bar{f}'(s)] = L^{-1}\left(\dfrac{1}{s+b}\right) - L^{-1}\left(\dfrac{1}{s+a}\right)$

ie., $\quad t f(t) = e^{-bt} - e^{-at}$

Thus $\quad \boldsymbol{f(t) = \dfrac{e^{-bt} - e^{-at}}{t}}$

---

39. Let $\bar{f}(s) = \log\left(1 - \dfrac{a^2}{s^2}\right) = \log\left(\dfrac{s^2-a^2}{s^2}\right)$

ie., $\quad \bar{f}(s) = \log(s^2 - a^2) - 2\log s$

$\therefore \quad -\bar{f}'(s) = -\left\{\dfrac{1}{s^2-a^2}\cdot 2s - \dfrac{2}{s}\right\} = 2\left(\dfrac{1}{s} - \dfrac{s}{s^2-a^2}\right)$

Now $\quad L^{-1}[-\bar{f}'(s)] = 2\left\{L^{-1}\left(\dfrac{1}{s}\right) - L^{-1}\left(\dfrac{s}{s^2-a^2}\right)\right\}$

ie., $\quad t f(t) = 2(1 - \cos h\, at)$

Thus $\quad \boldsymbol{f(t) = \dfrac{2(1-\cos h\, at)}{t}}$

---

40. Let $\bar{f}(s) = \cot^{-1}(s/a)$.

Differentiate w.r.t s and multiply with -1

$\therefore \quad \bar{f}'(s) = \dfrac{-1}{1+(s/a)^2}\cdot\dfrac{1}{a}$ and $-\bar{f}'(s) = \dfrac{a}{a^2+s^2}$

Taking inverse, $\quad L^{-1}[-\bar{f}'(s)] = L^{-1}\left(\dfrac{a}{s^2+a^2}\right)$

ie., $\quad t f(t) = \sin at$

Thus $\quad \boldsymbol{f(t) = \dfrac{\sin at}{t}}$

---

41. Let $\overline{f}(s) = \log \sqrt{s^2+1/s^2+4}$

$$= \frac{1}{2}\left\{ \log(s^2+1) - \log(s^2+4) \right\}$$

$\therefore \quad -\overline{f}'(s) = -\frac{1}{2}\left\{ \frac{2s}{s^2+1} - \frac{2s}{s^2+4} \right\} = \frac{s}{s^2+4} - \frac{s}{s^2+1}$

Now $\quad L^{-1}[-\overline{f}'(s)] = L^{-1}\left(\frac{s}{s^2+2^2}\right) - L^{-1}\left(\frac{s}{s^2+1^2}\right)$

ie., $\quad t f(t) = \cos 2t - \cos t$

Thus $\quad f(t) = \dfrac{\cos 2t - \cos t}{t}$

---

42. Let $\overline{f}(s) = \log\left[\frac{s^2+4}{s(s+4)(s-4)}\right]$

ie., $\quad \overline{f}(s) = \log(s^2+4) - \log s - \log(s+4) - \log(s-4)$

$\therefore \quad -\overline{f}'(s) = -\left\{ \frac{2s}{s^2+4} - \frac{1}{s} - \frac{1}{s+4} - \frac{1}{s-4} \right\}$

Now $\quad L^{-1}[-\overline{f}'(s)] = L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s+4}\right) + L^{-1}\left(\frac{1}{s-4}\right) - 2L^{-1}\left(\frac{s}{s^2+4}\right)$

ie., $\quad t f(t) = 1 + e^{-4t} + e^{4t} - 2\cos 2t = 1 + 2\cosh 4t - 2\cos 2t$

Thus $\quad f(t) = \dfrac{1 + 2(\cosh 4t - \cos 2t)}{t}$

---

43. Let $\overline{f}(s) = \tan^{-1}(2/s^2)$

$\therefore \quad \overline{f}'(s) = \frac{1}{1+(4/s^4)} \cdot \frac{-4}{s^3} = \frac{-4s}{s^4+4}$

Hence $\quad L^{-1}[-\overline{f}'(s)] = L^{-1}\left[\frac{4s}{s^4+4}\right]$

ie., $\quad t f(t) = L^{-1}\left[\frac{4s}{s^4+4}\right] \quad \ldots (1)$

Now $\quad s^4 + 4 = (s^2+2)^2 - 4s^2 = (s^2+2+2s)(s^2+2-2s)$

Also $\quad 4s = (s^2+2+2s) - (s^2+2-2s)$

Hence $\dfrac{4s}{s^4+4} = \dfrac{(s^2+2+2s)-(s^2+2-2s)}{(s^2+2+2s)(s^2+2-2s)}$

$$= \frac{1}{s^2+2-2s} - \frac{1}{s^2+2+2s}$$

$$\therefore \quad L^{-1}\left[\frac{4s}{s^4+4}\right] = L^{-1}\left[\frac{1}{s^2-2s+2}\right] - L^{-1}\left[\frac{1}{s^2+2s+2}\right]$$

Using (1) in L.H.S we have,

$$tf(t) = L^{-1}\left[\frac{1}{(s-1)^2+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

ie., $\quad tf(t) = e^t L^{-1}\left[\dfrac{1}{s^2+1}\right] - e^{-t} L^{-1}\left[\dfrac{1}{s^2+1}\right]$

ie., $\quad tf(t) = e^t \sin t - e^{-t}\sin t = \sin t\,(e^t - e^{-t})$

ie., $\quad tf(t) = \sin t \cdot 2\sinh t$

Thus $\quad \mathbf{f(t) = \dfrac{2\sin t\ \sinh t}{t}}$

---

44. Let $\bar{f}(s) = s\log\left(\dfrac{s+4}{s-4}\right)$

ie., $\quad \bar{f}(s) = s[\log(s+4) - \log(s-4)]$

$$\therefore \quad \bar{f}'(s) = \frac{s}{s+4} - \frac{s}{s-4} + [\log(s+4) - \log(s-4)]$$

We need to differentiate w.r.t s again,

$$\therefore \quad \bar{f}''(s) = \frac{4}{(s+4)^2} - \frac{(-4)}{(s-4)^2} + \frac{1}{s+4} - \frac{1}{s-4}$$

Now, $\quad L^{-1}[\bar{f}''(s)] = 4\left\{L^{-1}\left[\dfrac{1}{(s+4)^2}\right] + L^{-1}\left[\dfrac{1}{(s-4)^2}\right]\right\}$

$$+ L^{-1}\left[\frac{1}{s+4}\right] - L^{-1}\left[\frac{1}{s-4}\right]$$

ie., $\quad t^2 f(t) = 4\left\{e^{-4t} L^{-1}\left(\dfrac{1}{s^2}\right) + e^{4t} L^{-1}\left(\dfrac{1}{s^2}\right)\right\} + e^{-4t} - e^{4t}$

$$= 4(e^{-4t}\,t + e^{4t}\cdot t) - 2\sinh 4t$$

$$t^2 f(t) = 8t\cosh 4t - 2\sinh 4t$$

Thus $\quad \mathbf{f(t) = \dfrac{2(4t\cosh 4t - \sinh 4t)}{t^2}}$

---

45. Let $\bar{f}(s) = \cot^{-1}\left(\frac{s+a}{b}\right)$

$$\therefore \quad \bar{f}'(s) = \frac{-1}{1+\frac{(s+a)^2}{b^2}} \cdot \frac{1}{b} = \frac{-b}{b^2+(s+a)^2}$$

Now, $L^{-1}[-\bar{f}'(s)] = L^{-1}\left[\frac{b}{(s+a)^2+b^2}\right] = e^{-at}L^{-t}\left[\frac{b}{s^2+b^2}\right]$

ie., $$tf(t) = e^{-at}\sin bt$$

Thus $f(t) = \dfrac{e^{-at}\sin bt}{t}$

Find the inverse Laplace transform of the following functions

46. $\dfrac{s}{(s^2+a^2)^2}$

47. $\dfrac{s}{(s^2-a^2)^2}$

48. $\dfrac{s^2-a^2}{(s^2+a^2)^2}$

49. $\dfrac{s^2+a^2}{(s^2-a^2)^2}$

[*Here $\bar{f}(s)$ involves algebraic functions and we judiciously assume or compute an inverse transform based on the given $\bar{f}(s)$ and use $L^{-1}[-\bar{f}'(s)] = tf(t)$ to obtain $f(t)$*]

46. We have $L^{-1}\left[\dfrac{1}{s^2+a^2}\right] = \dfrac{1}{a}\sin at$ . . . (1)

Using the property that $L^{-1}[-\bar{f}'(s)] = tf(t)$, (1) becomes

$$L^{-1}\left[-\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)\right] = t\cdot\frac{1}{a}\sin at$$

ie., $$L^{-1}\left[\frac{2s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{a}$$

Thus $$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a}$$

---

47. We have $L^{-1}\left[\dfrac{1}{s^2-a^2}\right] = \dfrac{\sinh at}{a}$

$\therefore \quad L^{-1}\left[\dfrac{-d}{ds}\left(\dfrac{1}{s^2-a^2}\right)\right] = t \cdot \dfrac{\sinh at}{a}$

ie., $\quad L^{-1}\left[\dfrac{2s}{(s^2-a^2)^2}\right] = \dfrac{t\sinh at}{a}$

Thus $\quad \mathbf{L^{-1}\left[\dfrac{s}{(s^2-a^2)^2}\right] = \dfrac{t\sinh at}{2a}}$

---

48. We have $L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at$

$\therefore \quad L^{-1}\left[\dfrac{-d}{ds}\left(\dfrac{s}{s^2+a^2}\right)\right] = t\cos at$

ie., $\quad L^{-1}\left[\dfrac{-(s^2+a^2)+s\cdot 2s}{(s^2+a^2)^2}\right] = t\cos at$

Thus $\quad \mathbf{L^{-1}\left[\dfrac{s^2-a^2}{(s^2+a^2)^2}\right] = t\cos at}$

---

49. We have $L^{-1}\left[\dfrac{s}{s^2-a^2}\right] = \cosh at$

$\therefore \quad L^{-1}\left[\dfrac{-d}{ds}\left(\dfrac{s}{s^2-a^2}\right)\right] = t\cosh at$

ie., $\quad L^{-1}\left[\dfrac{-(s^2-a^2)+2s^2}{(s^2-a^2)^2}\right] = t\cosh at$

Thus $\quad \mathbf{L^{-1}\left[\dfrac{s^2+a^2}{(s^2-a^2)^2}\right] = t\cosh at}$

---

50. ***Find (a)*** $\mathbf{L^{-1}\left[\dfrac{1}{s}\sin\left(\dfrac{1}{s}\right)\right]}$ ***(b)*** $\mathbf{L^{-1}\left[\dfrac{1}{s}\cos\left(\dfrac{1}{s}\right)\right]}$

>> Inverse transforms of $\sin(1/s)$ and $\cos(1/s)$ is not readily available. We consider the expansion of $\sin x$ and $\cos x$ in the neighbourhood of the origin (*ie.,* $x = 0$) which are given by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \ ; \ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Replacing x by $1/s$ where we shall assume that s is large in which case $x = 1/s \to 0$

$$\therefore \quad \sin\left(\frac{1}{s}\right) = \frac{1}{s} - \frac{1}{3!}\frac{1}{s^3} + \frac{1}{5!}\frac{1}{s^5} - \cdots \quad \text{and} \quad \cos\left(\frac{1}{s}\right) = 1 - \frac{1}{2!}\frac{1}{s^2} + \frac{1}{4!}\frac{1}{s^4} - \cdots$$

Hence $\frac{1}{s}\sin\left(\frac{1}{s}\right) = \frac{1}{s^2} - \frac{1}{3!}\frac{1}{s^4} + \frac{1}{5!}\frac{1}{s^6} - \cdots$ and $\frac{1}{s}\cos\left(\frac{1}{s}\right) = \frac{1}{s} - \frac{1}{2!}\frac{1}{s^3} + \frac{1}{4!}\frac{1}{s^5} - \cdots$

$$\therefore \quad L^{-1}\left[\frac{1}{s}\sin\left(\frac{1}{s}\right)\right] = \frac{t^1}{1!} - \frac{1}{3!}\frac{t^3}{3!} + \frac{1}{5!}\frac{t^5}{5!} - \cdots$$

and $$L^{-1}\left[\frac{1}{s}\cos\left(\frac{1}{s}\right)\right] = 1 - \frac{1}{2!}\frac{t^2}{2!} + \frac{1}{4!}\frac{t^4}{4!} - \cdots$$

Thus $$L^{-1}\left[\frac{1}{s}\sin\left(\frac{1}{s}\right)\right] = t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \cdots$$

and $$L^{-1}\left[\frac{1}{s}\cos\left(\frac{1}{s}\right)\right] = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \cdots$$

EXERCISES

Find the inverse Laplace transform of the following functions.

I

1. $\dfrac{s^2 - 3s + 4}{s^3}$

2. $\dfrac{(s+2)^3}{s^6}$

3. $\dfrac{1}{3s-2} + \dfrac{4}{5s+1} + \dfrac{1}{s\sqrt{s}}$

4. $\dfrac{2s-5}{4s^2+25} + \dfrac{4s-18}{9-s^2}$

5. $\dfrac{(24 - 30\sqrt{s})\,e^{-s}}{s^4}$

II

6. $\dfrac{s-3}{s^2 - 6s + 13}$

7. $\dfrac{2s-3}{s^2 - 2s + 5}$

8. $\dfrac{s+3}{4s^2 + 4s + 9}$

9. $\dfrac{(3s+1)\,e^{s}}{(s+1)^4}$

10. $\dfrac{s\,e^{-2s}}{s^2 + 8s + 16}$

III ✓

11. $\dfrac{2s^2 - 6s + 5}{(s-1)(s-2)(s-3)}$

12. $\dfrac{4s+3}{(s-1)^2(s+2)}$

13. $\dfrac{3s-1}{(s-3)(s^2+4)}$

14. $\dfrac{(s^2+6)e^{2s}}{(s^2+1)(s^2+4)}$

15. $\dfrac{s}{s^4+64}$

16. $\dfrac{s^2 e^{3s}}{s^4+4}$

IV

17. $\dfrac{e^{-3s}}{(s-4)^2}+\dfrac{e^{-5s}}{(s-2)^4}$

18. $\dfrac{s e^{-\pi s}}{s^2+16}+\dfrac{e^{-\pi s}}{s^2+9}$

19. $\dfrac{3e^{-3s}}{s}-\dfrac{e^{-s}}{s^2}$

20. $\dfrac{5e^{-3s}}{s}-\dfrac{e^{-s}}{s}$

V

21. $\log\left(1+\dfrac{a^2}{s^2}\right)$

22. $\log\left[\dfrac{s^2+1}{s(s+1)}\right]$

23. $\log\left[\dfrac{s^2+4}{(s-4)^2}\right]$

24. $\dfrac{s+2}{(s^2+4s+5)^2}$

25. $\tan^{-1}(a/s)$

ANSWERS

I

1. $1-3t+2t^2$

2. $\dfrac{1}{30}(15t^2+30t^3+15t^4+2t^5)$

3. $\dfrac{1}{3}e^{2t/3}+\dfrac{4}{5}e^{-t/5}+2\sqrt{t/\pi}$

4. $\dfrac{1}{2}\{\cos(5t/2)-\sin(5t/2)\}-4\cosh 3t+6\sinh 3t$

5. $\{4(t-1)^3-(16/\sqrt{\pi})(t-1)^{5/2}\}u(t-1)$

II

6. $e^{3t}\cos 2t$

7. $e^t/2\cdot(4\cos 2t-\sin 2t)$

8. $\dfrac{e^{-t/2}}{8\sqrt{2}}\{2\sqrt{2}\cos(\sqrt{2}\,t)+5\sin(\sqrt{2}\,t)\}$

9. $e^{-(t+1)}\left\{\dfrac{3(t+1)^2}{2}-\dfrac{(t+1)^3}{3}\right\}u(t+1)$

10. $e^{-4(t-2)}\{1-4(t-2)\}u(t-2)$

III

11. $\frac{e^t}{2} - e^{2t} + \frac{5}{2} e^{3t}$

12. $\frac{5}{9} e^t + \frac{7}{3} t e^t - \frac{5}{9} e^{-2t}$

13. $\frac{8}{13} e^{3t} - \frac{8}{13} \cos 2t + \frac{15}{26} \sin 2t$

14. $\frac{1}{3}\{5 \sin(t+2) - \sin 2(t+2)\} u(t+2)$

15. $\frac{\sin 2t \sin h\, 2t}{8}$

16. $\frac{1}{2}\{\sin h\,(t-3) \cos(t-3) + \cos h\,(t-3) \sin(t-3)\} u(t-3)$

IV

17. $(t-3) e^{4(t-3)} u(t-3) + \frac{(t-5)^3 e^{2(t-5)} u(t-5)}{6}$

18. $\left\{\cos 4t - \frac{\sin 3t}{3}\right\} u(t-\pi)$

19. $3 \cdot u(t-3) - (t-1) u(t-1)$

20. $5u(t-3) - u(t-1)$

V

21. $\frac{2(1-\cos at)}{t}$

22. $\frac{1+e^{-t}-2\cos t}{t}$

23. $\frac{2(e^{4t}-\cos 2t)}{t}$

24. $\frac{t e^{-2t} \sin t}{2}$

25. $\frac{\sin at}{t}$

8.3 Convolution

Definition : The convolution of two functions $f(t)$ and $g(t)$ usually denoted by $f(t)*g(t)$ is defined in the form of an integral as follows.

$$f(t)*g(t) = \int_{u=0}^{t} f(u)\,g(t-u)\,du$$

Property : $f(t)*g(t) = g(t)*f(t)$

*That is to say that the convolution operation * is commutative.*

Proof : We have fromthe definition of convolution

$$f(t)*g(t) = \int_{u=0}^{t} f(u)\,g(t-u)\,du \qquad \ldots(1)$$

Put $t-u = v$ in (1) $\therefore\ -du = dv$ or $du = -dv$

If $u = 0,\ v = t$; If $u = t,\ v = 0$. Also $t-v = u$

Hence $f(t)*g(t) = \int_{v=t}^{0} f(t-v)\,g(v)\,(-dv)$

ie., $f(t)*g(t) = \int_{v=0}^{t} f(t-v)\,g(v)\,dv$ or $\int_{v=0}^{t} g(v)\,f(t-v)\,dv$

Comparing the R.H.S with (1) we have,

$$\boldsymbol{f(t)*g(t) = g(t)*f(t)}$$

This proves that the operation * is commutative.

8.31 Convolution theorem

If $L^{-1}[\bar{f}(s)] = f(t)$ *and* $L^{-1}[\bar{g}(s)] = g(t)$ *then*

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)\,g(t-u)\,du$$

Proof : We shall show that

$$L\left[\int_{u=0}^{t} f(u)\,g(t-u)\,du\right] = \bar{f}(s)\cdot\bar{g}(s)$$

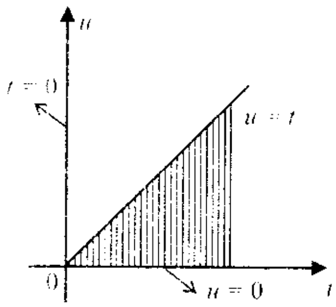
We have L.H.S by the definition

$$= \int_{t=0}^{\infty} e^{-st} \left[\int_{u=0}^{t} f(u)\, g(t-u)\, du \right] dt$$

$$= \int_{t=0}^{\infty} \int_{u=0}^{t} e^{-st} f(u)\, g(t-u)\, du\, dt \qquad \ldots (1)$$

We shall change the order of integration in respect of this double integral.

Existing region :



$t = 0$ to ∞ (Horizontal strip)

$u = 0$ to t (Vertical strip)

On changing the order :

$u = 0$ to ∞ (Vertical strip)

$t = u$ to ∞ (Horizontal strip)

On changing the order of integration, (1) becomes

$$\int_{u=0}^{\infty} \int_{t=u}^{\infty} e^{-st} f(u)\, g(t-u)\, dt\, du \qquad \ldots (2)$$

Now, let us put $t - u = v$ where u is fixed $\therefore\ dt = dv$

If $t = u$, $v = 0$; If $t = \infty$ $v = \infty$ and hence (2) becomes

$$\int_{u=0}^{\infty} \int_{v=0}^{\infty} e^{-s(u+v)} f(u)\, g(v)\, dv\, du$$

$$= \left[\int_{u=0}^{\infty} e^{-su} f(u)\, du \right] \cdot \left[\int_{v=0}^{\infty} e^{-sv} g(v)\, dv \right]$$

$$= \bar{f}(s) \cdot \bar{g}(s) = \text{R.H.S}$$

Hence we have proved that

$$L \left[\int_{u=0}^{t} f(u)\, g(t-u)\, du \right] = \bar{f}(s) \cdot \bar{g}(s)$$

Thus $\quad L^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = \int_{u=0}^{t} f(u)\, g(t-u)\, du$

This proves the convolution theorem.

Remarks :

1. *The integral in R.H.S is the convolution of the functions* $f(t)$ *and* $g(t)$ *denoted by* $f(t)*g(t)$

Thus the convolution theorem can be put in the form

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = f(t)*g(t)$$

2. *Since we have proved that* $f(t)*g(t) = g(t)*f(t)$,

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du = \int_{u=0}^{t} f(t-u)g(u)\,du$$

The integral in either of the forms is called as the convolution integral.

WORKED PROBLEMS

Type-1 *Verification of the convolution theorem* -

Working procedure for problems

We need to verify the theorem in respect of the two given functions $f(t)$ and $g(t)$

- We find $\bar{f}(s) = L[f(t)]$ and $\bar{g}(s) = L[g(t)]$
- We evaluate $f(t)*g(t) = \int_{0}^{t} f(u)g(t-u)\,du$
- We find $L[f(t)*g(t)]$.
- If $L[f(t)*g(t)] = \bar{f}(s)\cdot\bar{g}(s)$ then we can conclude that the theorem is verified.

Verify convolution theorem for the following pair of functions

51. $f(t) = t$ *and* $g(t) = \cos t$

52. $f(t) = \sin t$ *and* $g(t) = e^{-t}$

53. $f(t) = \cos at$ *and* $g(t) = \cos bt$

54. $f(t) = t$ *and* $g(t) = te^{-t}$

51. $\bar{f}(s) = L[f(t)] = L(t) = \dfrac{1}{s^2}$

$$\bar{g}(s) = L[g(t)] = L(\cos t) = \frac{s}{s^2+1}$$

$$f(t)*g(t) = \int_{u=0}^{t} f(u)g(t-u)\,du = \int_{0}^{t} u\cos(t-u)\,du$$

Applying Bernoulli's rule to the integral we get,

$$= [\, u \cdot -\sin(t-u)\,]_{u=0}^{t} - [\, 1 \cdot -\cos(t-u)\,]_{u=0}^{t}$$

$$= -(0-0) + (1-\cos t) = 1-\cos t$$

$$\therefore \quad L[f(t)*g(t)] = L(1) - L(\cos t) = \frac{1}{s} - \frac{s}{s^2+1} = \frac{1}{s(s^2+1)}$$

Also $\bar{f}(s)\cdot\bar{g}(s) = \frac{1}{s^2}\cdot\frac{s}{s^2+1} = \frac{1}{s(s^2+1)}$

Thus $L[f(t)*g(t)] = \bar{f}(s)\cdot\bar{g}(s)$. The theorem is verified.

---

52. $\bar{f}(s) = L(\sin t) = \frac{1}{s^2+1}, \ \bar{g}(s) = L(e^{-t}) = \frac{1}{s+1}$

$$f(t)*g(t) = \int_{u=0}^{t} f(u)\,g(t-u)\,du = \int_{u=0}^{t} \sin u \cdot e^{-(t-u)}\,du$$

$$= e^{-t}\int_{u=0}^{t} e^{u}\sin u\,du$$

$$= e^{-t}\left[\frac{e^{u}}{1+1}(\sin u - \cos u)\right]_{u=0}^{t}$$

$$= \frac{e^{-t}}{2}[\, e^{t}(\sin t - \cos t) + 1\,] = \frac{\sin t - \cos t}{2} + \frac{e^{-t}}{2}$$

$$\therefore \quad L[f(t)*g(t)] = \frac{1}{2}\left[\frac{1}{s^2+1} - \frac{s}{s^2+1} + \frac{1}{s+1}\right]$$

$$= \frac{1}{2}\left[\frac{(s+1) - s(s+1) + (s^2+1)}{(s+1)(s^2+1)}\right] = \frac{1}{(s+1)(s^2+1)}$$

Also $\bar{f}(s)\cdot\bar{g}(s) = \frac{1}{(s^2+1)(s+1)}$

Thus $L[f(t)*g(t)] = \bar{f}(s)\cdot\bar{g}(s)$. The theorem is verified.

---

53. $\bar{f}(s) = L(\cos at) = \frac{s}{s^2+a^2}$; $\bar{g}(s) = L(\cos bt) = \frac{s}{s^2+b^2}$

$$f(t)*g(t) = \int_{u=0}^{t} f(u)\,g(t-u)\,du = \int_{u=0}^{t} \cos au \cdot \cos(bt-bu)\,du$$

$$f(t)*g(t) = \frac{1}{2}\int_{u=0}^{t} [\cos(au+bt-bu) + \cos(au-bt+bu)]\,du$$

$$= \frac{1}{2}\left[\frac{\sin(au+bt-bu)}{a-b} + \frac{\sin(au-bt+bu)}{a+b}\right]_{u=0}^{t}$$

$$= \frac{1}{2}\left[\frac{1}{a-b}\{\sin at - \sin bt\} + \frac{1}{a+b}\{\sin at + \sin bt\}\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{1}{a-b} + \frac{1}{a+b}\right) + \sin bt\left(\frac{1}{a+b} - \frac{1}{a-b}\right)\right]$$

$$= \frac{1}{2}\left[\sin at \cdot \frac{2a}{a^2-b^2} + \sin bt \cdot \frac{-2b}{a^2-b^2}\right]$$

$$= \frac{1}{a^2-b^2}(a\sin at - b\sin bt),\ a \neq b$$

$$\therefore \quad L[f(t)*g(t)] = \frac{1}{a^2-b^2}\left[a \cdot \frac{a}{s^2+a^2} - b \cdot \frac{b}{s^2+b^2}\right]$$

$$= \frac{1}{a^2-b^2}\left[\frac{a^2(s^2+b^2) - b^2(s^2+a^2)}{(s^2+a^2)(s^2+b^2)}\right]$$

ie., $$L[f(t)*g(t)] = \frac{1}{a^2-b^2} \cdot \frac{s^2(a^2-b^2)}{(s^2+a^2)(s^2+b^2)} = \frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

Also $$\bar{f}(s)\cdot\bar{g}(s) = \frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

Thus $L[f(t)*g(t)] = \bar{f}(s)\cdot\bar{g}(s)$. The theorem is verified.

54. $\overline{f}(s) = L(t) = \frac{1}{s^2}$, $\overline{g}(s) = L(e^{-t} \cdot t) = \frac{1}{(s+1)^2}$

$$f(t) * g(t) = \int_{u=0}^{t} f(u) g(t-u)\, du = \int_{u=0}^{t} u\, e^{-(t-u)} (t-u)\, du$$

$$= e^{-t} \int_{u=0}^{t} (tu - u^2) e^{u}\, du$$

$$= e^{-t} [\, (tu - u^2) e^{u} - (t - 2u) e^{u} + (-2) e^{u} \,]_{u=0}^{t}$$

$$= e^{-t} [\, (0-0) - (-t e^{t} - t) - 2(e^{t} - 1) \,]$$

$$= t + t e^{-t} - 2 + 2 e^{-t}$$

$$L[f(t) * g(t)] = \frac{1}{s^2} + \frac{1}{(s+1)^2} - \frac{2}{s} + \frac{2}{s+1}$$

$$= \frac{(s+1)^2 + s^2 - 2s(s+1)^2 + 2s^2(s+1)}{s^2 (s+1)^2} = \frac{1}{s^2 (s+1)^2}$$

Thus $L[f(t) * g(t)] = \overline{f}(s) \cdot \overline{g}(s)$. The theorem is verified.

Type-2 *Computation of the inverse transform by using convolution theorem*

Working procedure for problems.

- The given function is expressed as the product of two functions say $\overline{f}(s)$ and $\overline{g}(s)$
- We find $L^{-1}[\overline{f}(s)] = f(t)$ and $L^{-1}[\overline{g}(s)] = g(t)$
- We apply the convolution theorem in one of the form :

  $$L^{-1}[\overline{f}(s) \cdot \overline{g}(s)] = \int_{u=0}^{t} f(u) g(t-u)\, du$$

- We evaluate the convolution integral to obtain the required inverse.

Using convolution theorem obtain the inverse Laplace transform of the following functions

55. $\dfrac{1}{s(s^2+a^2)}$

56. $\dfrac{s}{(s^2+a^2)^2}$

57. $\dfrac{s^2}{(s^2+a^2)^2}$

58. $\dfrac{1}{(s^2+a^2)^2}$

59. $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

60. $\dfrac{1}{(s-1)(s^2+1)}$

61. $\dfrac{1}{s^2(s+1)^2}$

62. $\dfrac{s+2}{(s^2+4s+5)^2}$

63. $\dfrac{1}{(s^2+4s+13)^2}$

64. $\dfrac{4s+5}{(s-1)^2(s+2)}$

65. $\dfrac{1}{(s^2+a^2)^3}$

55. Let $\overline{f}(s) = \dfrac{1}{s}$; $\overline{g}(s) = \dfrac{1}{s^2+a^2}$

Taking inverse,

$$f(t) = L^{-1}\left[\frac{1}{s}\right] = 1 \; ; \; g(t) = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$$

We have convolution theorem,

$$L^{-1}[\overline{f}(s)\cdot\overline{g}(s)] = \int_{u=0}^{t} f(u)\,g(t-u)\,du$$

$$\therefore \quad L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_{u=0}^{t} 1\cdot\frac{\sin(at-au)}{a}\,du$$

$$= \left[\frac{\cos(at-au)}{a^2}\right]_{u=0}^{t} = \frac{1}{a^2}(1-\cos at)$$

Thus $L^{-1}\left[\dfrac{1}{s(s^2+a^2)}\right] = \dfrac{1}{a^2}(1-\cos at)$

---

56. Let $\bar{f}(s) = \frac{1}{s^2 + a^2}$; $\bar{g}(s) = \frac{s}{s^2 + a^2}$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = \frac{\sin at}{a}$; $g(t) = L^{-1}[\bar{g}(s)] = \cos at$

We have convolution theorem

$$L^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = \int_{u=0}^{t} f(u)\, g(t-u)\, du$$

$$\therefore \quad L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \int_{u=0}^{t} \frac{\sin au}{a} \cdot \cos(at - au)\, du$$

$$= \frac{1}{2a} \int_{u=0}^{t} [\sin(au + at - au) + \sin(au - at + au)]\, du$$

$$= \frac{1}{2a} \int_{u=0}^{t} [\sin at + \sin(2au - at)]\, du$$

$$= \frac{1}{2a}\left\{\sin at\, [u]_{u=0}^{t} - \left[\frac{\cos(2au - at)}{2a}\right]_{u=0}^{t}\right\}$$

$$= \frac{1}{2a}\left\{\sin at\,(t - 0) - \frac{1}{2a}(\cos at - \cos at)\right\} = \frac{t \sin at}{2a}$$

Thus $\quad L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t \sin at}{2a}$

---

57. Let $\bar{f}(s) = \frac{s}{s^2 + a^2} = \bar{g}(s)$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = \cos at = g(t)$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \int_{u=0}^{t} \cos au \; \cos(at - au)\, du$$

$$= \frac{1}{2} \int_{u=0}^{t} [\cos(au + at - au) + \cos(au - at + au)]\, du$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2}\int_{u=0}^{t}[\cos at + \cos(2au - at)]\,du$$

$$= \frac{1}{2}\left\{\cos at\,[u]_0^t + \left[\frac{\sin(2au - at)}{2a}\right]_0^t\right\}$$

$$= \frac{1}{2}\left\{\cos at\,(t-0) + \frac{1}{2a}(\sin at + \sin at)\right\} = \frac{1}{2}\left\{t\cos at + \frac{\sin at}{a}\right\}$$

Thus $\quad L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2a}(at\cos at + \sin at)$

---

58. Let $\bar{f}(s) = \frac{1}{s^2+a^2} = \bar{g}(s)$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = \frac{\sin at}{a} = g(t)$

Now by applying convolution theorem we have

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \int_{u=0}^{t}\frac{\sin au}{a}\cdot\frac{\sin(at-au)}{a}\,du$$

$$= \frac{1}{2a^2}\int_{u=0}^{t}[\cos(au - at + au) - \cos(au + at - au)]\,du$$

$$= \frac{1}{2a^2}\int_{u=0}^{t}[\cos(2au - at) - \cos at]\,du$$

$$= \frac{1}{2a^2}\left\{\left[\frac{\sin(2au-at)}{2a}\right]_0^t - \cos at\,[u]_0^t\right\}$$

$$= \frac{1}{2a^2}\left\{\frac{1}{2a}(\sin at + \sin at) - \cos at \cdot t\right\}$$

Thus $\quad L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)$

---

59. Let $\bar{f}(s) = \dfrac{s}{s^2 + a^2}$; $\bar{g}(s) = \dfrac{s}{s^2 + b^2}$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = \cos at$; $g(t) = L^{-1}[\bar{g}(s)] = \cos bt$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{s^2}{(s^2 + a^2)(s^2 + b^2)}\right] = \int_{u=0}^{t} \cos au \cdot \cos(bt - bu)\, du$$

[*Refer problem-53 for the integration process*]

Thus $\quad$ $$\mathbf{L^{-1}\left[\frac{s^2}{(s^2 + a^2)(s^2 + b^2)}\right] = \frac{a \sin at - b \sin bt}{a^2 - b^2},\ a \neq b}$$

---

60. Let $\bar{f}(s) = \dfrac{1}{s - 1}$; $\bar{g}(s) = \dfrac{1}{s^2 + 1}$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = e^t$; $g(t) = L^{-1}[\bar{g}(s)] = \sin t$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{(s - 1)(s^2 + 1)}\right] = \int_{u=0}^{t} e^u \cdot \sin(t - u)\, du$$

But $\displaystyle\int e^{ax} \sin(bx + c)\, dx = \frac{e^{ax}}{a^2 + b^2}[a \sin(bx + c) - b \cos(bx + c)]$

$$L^{-1}\left[\frac{1}{(s - 1)(s^2 + 1)}\right] = \left[\frac{e^u}{1 + 1}\{\sin(t - u) + \cos(t - u)\}\right]_0^t$$

$$= \frac{1}{2}\{e^t(0 + 1) - 1(\sin t + \cos t)\}$$

Thus $\quad$ $$\mathbf{L^{-1}\left[\frac{1}{(s - 1)(s^2 + 1)}\right] = \frac{1}{2}(e^t - \sin t - \cos t)}$$

---

61. Let $\bar{f}(s) = \dfrac{1}{s^2}$; $\bar{g}(s) = \dfrac{1}{(s + 1)^2}$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = t$; $g(t) = L^{-1}[\bar{g}(s)] = e^{-t}t.$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = \int_{u=0}^{t} u\, e^{-(t-u)}(t-u)\,du = e^{-t}\int_{u=0}^{t}(tu-u^2)\,e^{u}\,du$$

[*Refer Problem-54 for the integration process*]

Thus $\mathbf{L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = t+te^{-t}-2+2e^{-t} = 2(e^{-t}-1)+t(1+e^{-t})}$

---

62. Let $\bar{f}(s) = \dfrac{s+2}{s^2+4s+5}$; $\bar{g}(s) = \dfrac{1}{s^2+4s+5}$

$$\Rightarrow \quad f(t) = L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] \quad ; \quad g(t) = L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$f(t) = e^{-2t}L^{-1}\left[\frac{s}{s^2+1}\right] \quad ; \quad g(t) = e^{-2t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$\therefore \quad f(t) = e^{-2t}\cos t \quad ; \quad g(t) = e^{-2t}\sin t$$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right] = \int_{u=0}^{t} e^{-2u}\cos u\, e^{-2(t-u)}\sin(t-u)\,du$$

$$= e^{-2t}\int_{u=0}^{t}\sin(t-u)\cos u\,du$$

$$= \frac{e^{-2t}}{2}\int_{u=0}^{t}[\sin(t-u+u)+\sin(t-u-u)]\,du$$

$$= \frac{e^{-2t}}{2}\int_{u=0}^{t}[\sin t+\sin(t-2u)]\,du$$

$$= \frac{e^{-2t}}{2}\left\{\sin t\,[u]_0^t + \left.\frac{\cos(t-2u)}{2}\right|_{u=0}^{t}\right\}$$

$$= \frac{e^{-2t}}{2}\left\{t\sin t+\frac{1}{2}(\cos t-\cos t)\right\}$$

Thus $\mathbf{L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right] = \frac{e^{-2t}\,t\sin t}{2}}$

---

63. Let $\bar{f}(s) = \dfrac{1}{s^2+4s+13} = \bar{g}(s)$

$\Rightarrow \quad f(t) = L^{-1}\left[\dfrac{1}{(s+2)^2+3^2}\right] = g(t)$

ie., $\quad f(t) = e^{-2t} L^{-1}\left[\dfrac{1}{s^2+3^2}\right] = \dfrac{e^{-2t}\sin 3t}{3} = g(t)$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right] = \int_{u=0}^{t} \frac{e^{-2u}\sin 3u}{3} \cdot \frac{e^{-2(t-u)}\sin(3t-3u)}{3}\, du$$

$$= \frac{e^{-2t}}{9}\int_{u=0}^{t} \sin 3u \cdot \sin(3t-3u)\, du$$

$$= \frac{e^{-2t}}{18}\int_{u=0}^{t} [\cos(3u-3t+3u) - \cos(3u+3t-3u)]\, du$$

$$= \frac{e^{-2t}}{18}\int_{u=0}^{t} [\cos(6u-3t) - \cos 3t]\, du$$

$$= \frac{e^{-2t}}{18}\left\{\left[\frac{\sin(6u-3t)}{6}\right]_0^t - \cos 3t\,[u]_0^t\right\}$$

$$= \frac{e^{-2t}}{18}\left\{\frac{\sin 3t+\sin 3t}{6} - \cos 3t \cdot t\right\}$$

Thus $\quad L^{-1}\left[\dfrac{1}{(s^2+4s+13)^2}\right] = \dfrac{e^{-2t}}{54}(\sin 3t - 3t\cos 3t)$

---

64. Let $\bar{f}(s) = \dfrac{1}{s+2}$; $\bar{g}(s) = \dfrac{4s+5}{(s-1)^2}$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = e^{-2t}$

Also, $\quad g(t) = L^{-1}[\bar{g}(s)] = L^{-1}\left[\dfrac{4s+5}{(s-1)^2}\right] = L^{-1}\left[\dfrac{4(s-1)+9}{(s-1)^2}\right]$

$$g(t) = e^t L^{-1}\left[\frac{4s+9}{s^2}\right] = e^t(4+9t)$$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{s+2}\cdot\frac{4s+5}{(s-1)^2}\right] = \int_{u=0}^{t} e^{-2u}\cdot e^{(t-u)}[4+9(t-u)]\,du$$

$$= e^t\int_{u=0}^{t} e^{-3u}(4+9t-9u)\,du$$

$$= e^t\int_{u=0}^{t}(4+9t-9u)\,e^{-3u}\,du$$

Integrating R.H.S by parts we get,

$$L^{-1}\left[\frac{4s+5}{(s-1)^2(s+2)}\right] = e^t\left\{(4+9t-9u)\frac{e^{-3u}}{-3}-(-9)\frac{e^{-3u}}{9}\right\}_{u=0}^{t}$$

$$= e^t\left\{4\frac{e^{-3t}}{-3}-\frac{(4+9t)}{-3}+e^{-3t}-1\right\}$$

$$= e^t\left\{\frac{1}{3}-\frac{1}{3}e^{-3t}+3t\right\}$$

Thus $\quad L^{-1}\left[\dfrac{4s+5}{(s-1)^2(s+2)}\right] = \dfrac{1}{3}e^t-\dfrac{1}{3}e^{-2t}+3\,te^t$

---

65. Let $\bar{f}(s) = \dfrac{1}{s^2+a^2}$; $\bar{g}(s) = \dfrac{1}{(s^2+a^2)^2}$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = \dfrac{\sin at}{a}$; $g(t) = L^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right]$

ie., $\quad f(t) = \dfrac{\sin at}{a}$; $g(t) = \dfrac{1}{2a^3}(\sin at - at\cos at)$ *(Refer Ex-139)*

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{(s^2+a^2)}\cdot\frac{1}{(s^2+a^2)^2}\right]$$

$$= \int_{u=0}^{t}\frac{\sin au}{a}\cdot\frac{1}{2a^3}\,[\sin(at-au)-a(t-u)\cos(at-au)]\,du$$

$$= \frac{1}{2a^4}\int_{u=0}^{t}\sin au\cdot\sin(at-au)\,du$$

$$- \frac{1}{2a^4}\int_{u=0}^{t}[a(t-u)\sin au\cdot\cos(at-au)]\,du = I_1 - I_2\ (say)$$

Now $$I_1 = \frac{1}{2a^4}\int_{u=0}^{t}\sin au\cdot\sin(at-au)\,du$$

$$= \frac{1}{4a^4}\int_{u=0}^{t}[\cos(au-at+au)-\cos(au+at-au)]\,du$$

$$= \frac{1}{4a^4}\left\{\left[\frac{\sin(2au-at)}{2a}\right]_{u=0}^{t} - \cos at\,[u]_0^t\right\}$$

$$= \frac{1}{4a^4}\left\{\frac{\sin at}{a} - t\cos at\right\} = \frac{1}{4a^5}(\sin at - at\cos at) \quad \ldots(1)$$

Next $$I_2 = \frac{1}{2a^4}\int_{u=0}^{t}(at-au)\sin au\cdot\cos(at-au)\,du$$

$$= \frac{1}{2a^4}\cdot\frac{1}{2}\int_{u=0}^{t}(at-au)[\sin(au+at-au)+\sin(au-at+au)]\,du$$

$$= \frac{1}{4a^4}\int_{u=0}^{t}(at-au)\sin at\,du + \frac{1}{4a^4}\int_{u=0}^{t}(at-au)\sin(2au-at)\,du$$

$$= \frac{1}{4a^4}\left[\left(atu-\frac{au^2}{2}\right)\sin at\right]_{u=0}^{t}$$

$$+ \frac{1}{4a^4}\left[(at-au)\cdot\frac{-\cos(2au-at)}{2a} - (-a)\cdot\frac{-\sin(2au-at)}{4a^2}\right]_{u=0}^{t}$$

$$I_2 = \frac{1}{4a^4}\left(\frac{at^2}{2}\sin at\right) + \frac{1}{4a^4}\left\{\frac{-1}{2a}(-at\cos at) - \frac{1}{4a}(2\sin at)\right\}$$

$$I_2 = \frac{t^2 \sin at}{8a^3} + \frac{t\cos at}{8a^4} - \frac{\sin at}{8a^5} \qquad \ldots (2)$$

From (1) and (2) we have $I_1 - I_2$ given by

$$\frac{\sin at}{4a^5} - \frac{t\cos at}{4a^4} - \frac{t^2\sin at}{8a^3} - \frac{t\cos at}{8a^4} + \frac{\sin at}{8a^5}$$

$$= -\frac{t^2\sin at}{8a^3} - \frac{3t\cos at}{8a^4} + \frac{3\sin at}{8a^5}$$

Thus $L^{-1}\left[\dfrac{1}{(s^2+a^2)^3}\right] = \dfrac{1}{8a^5}\left[(3 - a^2t^2)\,\sin at - 3\,at\cos at\right]$

Remark .

It is advisible to remember the following three inverse Laplace transforms.

(i) $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right] = \dfrac{t\sin at}{2a}$

(ii) $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)^2}\right] = \dfrac{1}{2a}(\sin at + at\cos at)$

(iii) $L^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right] = \dfrac{1}{2a^3}(\sin at - at\cos at)$

Type-3 *Laplace transform of the convolution integral and solution of integral equations*

Working procedure for problems

- We can find the Laplace transform of the convolution integral by using the result

$$L\left[\int_0^t f(u)\,g(t-u)\,du\right] = \overline{f}(s)\cdot\overline{g}(s) = L\left[\int_0^t f(t-u)\,g(u)\,du\right]$$

in the appropriate form.

- Given an equation for $f(t)$ involving the convolution integral we first take Laplace transform on both sides.
- We evaluate the convolution integral and simplify to obtain $L[f(t)] = \overline{f}(s)$ as a function of s.
- Taking inverse we obtain $f(t)$

Find the Laplace transform of the following convolution integrals.

66. $\int_0^t (t-u)\sin 2u\, du$

67. $\int_0^t e^{-u}\sin(t-u)\, du$

68. $\int_0^t \sin a(t-u)\cos au\, du$

69. $\int_0^t (t-u)\, u\, e^{-2u}\, du$

66. To find $L\left[\int_0^t (t-u)\sin 2u\, du\right]$ we use the result

$$L\left[\int_0^t f(t-u)\, g(u)\, du\right] = \bar{f}(s)\cdot\bar{g}(s) \qquad \ldots (1)$$

By comparing, we have $f(t-u) = (t-u)$ and $g(u) = \sin 2u$

Equivalently, $f(t) = t$ and $g(t) = \sin 2t$

$\therefore \quad \bar{f}(s) = L(t) = \dfrac{1}{s^2}$ and $\bar{g}(s) = L(\sin 2t) = \dfrac{2}{s^2+4}$

Thus from (1) we get,,

$$L\left[\int_0^t (t-u)\sin 2u\, du\right] = \frac{1}{s^2}\cdot\frac{2}{s^2+4} = \frac{2}{s^2(s^2+4)}$$

---

67. To find $L\left[\int_0^t e^{-u}\sin(t-u)\, du\right]$ we use the result

$$L\left[\int_0^t f(u)\, g(t-u)\, du\right] = \bar{f}(s)\cdot\bar{g}(s) \qquad \ldots (1)$$

By comparing, we have, $f(u) = e^{-u}$ and $g(t-u) = \sin(t-u)$

Equivalently, $f(t) = e^{-t}$ and $g(t) = \sin t$

$\therefore \quad \bar{f}(s) = L(e^{-t}) = \dfrac{1}{s+1}$ and $\bar{g}(s) = L(\sin t) = \dfrac{1}{s^2+1}$.

Thus from (1) we get,

$$L\left[\int_0^t e^{-u} \sin(t-u)\, du\right] = \frac{1}{(s+1)(s^2+1)}$$

---

68. To find $L\left[\int_0^t \sin a(t-u) \cos au\, du\right]$ we use the result

$$L\left[\int_0^t f(t-u)\, g(u)\, du\right] = \bar{f}(s) \cdot \bar{g}(s) \qquad \ldots (1)$$

By comparing we have, $f(t-u) = \sin a(t-u)$ and $g(u) = \cos au$

Equivalently, $f(t) = \sin at$ and $g(t) = \cos at$

$\therefore \quad \bar{f}(s) = L(\sin at) = \dfrac{a}{s^2+a^2}$ and $\bar{g}(s) = L(\cos at) = \dfrac{s}{s^2+a^2}$

Thus from (1) we get,

$$L\left[\int_0^t \sin a(t-u) \cos au\, du\right] = \frac{as}{(s^2+a^2)^2}$$

---

69. To find $L\left[\int_0^t (t-u)\, u e^{-2u}\, du\right]$ we use the result

$$L\left[\int_0^t f(t-u)\, g(u)\, du\right] = \bar{f}(s) \cdot \bar{g}(s) \qquad \ldots (1)$$

By comparing we have, $f(t-u) = (t-u)$ and $g(u) = u e^{-2u}$

Equivalently, $f(t) = t$ and $g(t) = t e^{-2t}$

$\therefore \quad \bar{f}(s) = L(t) = \dfrac{1}{s^2}$ and $\bar{g}(s) = L(e^{-2t} \cdot t) = \dfrac{1}{(s+2)^2}$

Thus from (1) we get,

$$L\left[\int_0^t (t-u)\, u e^{-2u}\, du\right] = \frac{1}{s^2(s+2)^2}$$

---

70. *Find $f(t)$ from the equation* $f(t) = 1 + 2\int_0^t f(t-u)\, e^{-2u}\, du$

>> Taking Laplace transform on both sides we have,

$$L[f(t)] = L(1) + 2L\left[\int_0^t f(t-u)\, e^{-2u}\, du\right]$$

ie., $$\bar{f}(s) = \frac{1}{s} + 2L\left[\int_0^t f(t-u)\, e^{-2u}\, du\right]$$

ie., $$\bar{f}(s) = \frac{1}{s} + 2\bar{f}(s)\cdot\bar{g}(s) \qquad \ldots (1)$$

where $\bar{g}(s) = L[g(u)] = L(e^{-2u}) = \dfrac{1}{s+2}$

Hence (1) becomes

$$\bar{f}(s) = \frac{1}{s} + 2\bar{f}(s)\cdot\frac{1}{s+2}$$

or $$\bar{f}(s)\left[1 - \frac{2}{s+2}\right] = \frac{1}{s} \quad \text{or} \quad \bar{f}(s)\left[\frac{s}{s+2}\right] = \frac{1}{s}$$

$\therefore$ $$\bar{f}(s) = \frac{s+2}{s^2} = \frac{1}{s} + \frac{2}{s^2}$$

By taking inverse we have,

$$L^{-1}[\bar{f}(s)] = L^{-1}\left(\frac{1}{s}\right) + 2L^{-1}\left(\frac{1}{s^2}\right)$$

Thus $\mathbf{f(t) = 1 + 2t}$

---

71. *Solve the integral equation :* $f(t) = 1 + \int_0^t f(u)\sin(t-u)\, du$

>> Taking Laplace transform on bothsides we have,

$$L[f(t)] = L(1) + L\left[\int_0^t f(u)\sin(t-u)\, du\right]$$

ie., $$\bar{f}(s) = \frac{1}{s} + \bar{f}(s)\cdot\bar{g}(s) \qquad \ldots (1)$$

where $g(t-u) = \sin(t-u)$ or $g(t) = \sin t$ $\therefore$ $\bar{g}(s) = \dfrac{1}{s^2+1}$

Hence (1) becomes,

$$\bar{f}(s) = \frac{1}{s} + \bar{f}(s) \cdot \frac{1}{s^2+1} \quad \text{or} \quad \bar{f}(s)\left[1 - \frac{1}{s^2+1}\right] = \frac{1}{s}$$

ie., $\bar{f}(s) \cdot \dfrac{s^2}{s^2+1} = \dfrac{1}{s}$ or $\bar{f}(s) = \dfrac{s^2+1}{s^3} = \dfrac{1}{s} + \dfrac{1}{s^3}$

Now $L^{-1}[\bar{f}(s)] = L^{-1}\left(\dfrac{1}{s}\right) + L^{-1}\left(\dfrac{1}{s^3}\right)$

Thus $\boldsymbol{f(t) = 1 + \dfrac{t^2}{2}}$

---

72. *Find $f(t)$ from the equation $f(t) = \dfrac{t^2}{2} - \displaystyle\int_0^t f(u)(t-u)\,du$*

>> Taking Laplace transform on both sides we have,

$$L[f(t)] = L\left(\frac{t^2}{2}\right) - L\left[\int_0^t f(u)\,(t-u)\,du\right]$$

ie., $\bar{f}(s) = \dfrac{1}{s^3} - \bar{f}(s) \cdot \bar{g}(s)$ ... (1)

where $g(t-u) = (t-u)$ or $g(t) = t$ $\therefore$ $\bar{g}(s) = \dfrac{1}{s^2}$

Hence (1) becomes,

$$\bar{f}(s) = \frac{1}{s^3} - \frac{\bar{f}(s)}{s^2} \quad \text{or} \quad \bar{f}(s)\left[1 + \frac{1}{s^2}\right] = \frac{1}{s^3}$$

ie., $\bar{f}(s) \cdot \dfrac{s^2+1}{s^2} = \dfrac{1}{s^3}$ or $\bar{f}(s) = \dfrac{1}{s(s^2+1)}$

Now $L^{-1}[\bar{f}(s)] = f(t) = L^{-1}\left[\dfrac{1}{s(s^2+1)}\right]$

But $\dfrac{1}{s(s^2+1)} = \dfrac{1}{s} - \dfrac{s}{s^2+1}$ by partial fractions.

$\therefore$ $L^{-1}\left[\dfrac{1}{s(s^2+1)}\right] = L^{-1}\left(\dfrac{1}{s}\right) - L^{-1}\left(\dfrac{s}{s^2+1}\right)$

Thus $\boldsymbol{f(t) = 1 - \cos t = 2\sin^2(t/2)}$

---

73. *Solve the integral equation* : $f(t) = 4t^2 - \int_0^t f(t-u)\, e^{-u}\, du$

>> Taking Laplace transform on l oth sides we have,

$$L[f(t)] = 4L(t^2) - L\left[\int_0^t f(t-u)\, e^{-u}\, du\right]$$

ie., $$\bar{f}(s) = \frac{8}{s^3} - \bar{f}(s) \cdot \bar{g}(s) \qquad \ldots(1)$$

where $g(u) = e^{-u}$ and hence $\bar{g}(s) = L(e^{-u}) = \dfrac{1}{s+1}$

Now (1) becomes,

$$\bar{f}(s) = \frac{8}{s^3} - \frac{\bar{f}(s)}{s+1} \quad \text{or} \quad \bar{f}(s)\left[1 + \frac{1}{s+1}\right] = \frac{8}{s^3}$$

ie., $$\bar{f}(s) \cdot \frac{s+2}{s+1} = \frac{8}{s^3} \quad \text{or} \quad \bar{f}(s) = \frac{8(s+1)}{s^3(s+2)}$$

Further $L^{-1}[\bar{f}(s)] = f(t) = L^{-1}\left[\dfrac{8(s+1)}{s^3(s+2)}\right] \qquad \ldots(2)$

Let $$\frac{8s+8}{s^3(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+2}$$

or $$8s + 8 = As^2(s+2) + Bs(s+2) + C(s+2) + Ds^3 \qquad \ldots(3)$$

Put $s = 0$: $8 = C(2)$ $\therefore$ $C = 4$

Put $s = -2$: $-8 = D(-8)$ $\therefore$ $D = 1$

Equating the coefficients of s^3, s^2 separately on both sides of (3) we get,

$0 = A + D$ and $0 = 2A + B$. $\therefore$ $A = -1$ and $B = 2$

Now $$L^{-1}\left[\frac{8s+8}{s^3(s+2)}\right] = -L^{-1}\left(\frac{1}{s}\right) + 2L^{-1}\left(\frac{1}{s^2}\right) + 4L^{-1}\left(\frac{1}{s^3}\right) + L^{-1}\left(\frac{1}{s+2}\right)$$

Thus $\boldsymbol{f(t) = -1 + 2t + 2t^2 + e^{-2t}}$

8.4 Laplace transform of the derivatives

We derive an expression for $L[y'(t)]$ and hence deduce the expressions for $L[y''(t)]$, $L[y'''(t)]\ldots$

Further we use the principle of mathematical induction to establish the result for $L[y^{(n)}(t)]$

$$L[y'(t)] = \int_0^\infty e^{-st} y'(t)\,dt$$

Integrating by parts we have,

$$L[y'(t)] = \left[e^{-st} y(t)\right]_{t=0}^{\infty} - \int_0^\infty y(t)\cdot e^{-st}(-s)\,dt$$

$$= [0 - 1\cdot y(0)] + s\int_0^\infty e^{-st} y(t)\,dt$$

$$= -y(0) + s\,L[y(t)]$$

Thus $\quad L[y'(t)] = s\,L[y(t)] - y(0) \qquad \ldots(1)$

Now $L[y''(t)] = L\left[\{y'(t)\}'\right]$ and applying (1) we have,

$$= s\,L[y'(t)] - y'(0)$$

ie., $\quad = s\{s\,L[y(t)] - y(0)\} - y'(0)$ by using (1).

$\therefore \quad L[y''(t)] = s^2 L[y(t)] - s\,y(0) - y'(0) \qquad \ldots(2)$

Also $\quad L[y'''(t)] = s^3 L[y(t)] - s^2 y(0) - s\,y'(0) - y''(0) \qquad \ldots(3)$

In general we shall show that

$$L[y^{(n)}(t)] = s^n L[y(t)] - s^{n-1} y(0) - s^{n-2} y'(0) - \cdots - y^{(n-1)}(0)$$

The result is established by induction.

When $n = 1$ we have

$$L[y'(t)] = s^1 L[y(t)] - s^0 y(0) = s\,L[y(t)] - y(0) \qquad \ldots(i)$$

Comparing with (1) we conclude that the result is true for $n = 1$.
Let us assume the result to be true for $n = k$.

$$L[y^{(k)}(t)] = s^k L[y(t)] - s^{k-1} y(0) - s^{k-2} y'(0) - \cdots - y^{(k-1)}(0) \qquad \ldots(ii)$$

$L[y^{(k+1)}(t)] = L\left[\{y^k(t)\}'\right] = s\,L[y^{(k)}(t)] - y^{(k)}(0)$ by (i).

Using (ii) in the R.H.S for $L[y^{(k)}(t)]$ we have,

$$L[y^{(k+1)}(t)] = s\left\{s^k L[y(t)] - s^{k-1}y(0) - s^{k-2}y'(0) - \cdots - y^{(k-1)}(0)\right\} - y^{(k)}(0)$$

ie., $= s^{k+1}L[y(t)] - s^k y(0) - s^{k-1}y'(0) - \ldots - sy^{(k-1)}(0) - y^{(k)}(0)$

ie., $$L[y^{(k+1)}(t)] = s^{k+1}L[y(t)] - s^{(k+1)-1}y(0) - s^{(k+1)-2}y'(0) - \ldots - y^{(k+1)-1}(0) \qquad \ldots \text{(iii)}$$

Comparing (ii) and (iii) we conclude that the result is true for $n = k+1$. Thus by the principle of induction the result is true for all positive integral values of n.

8.41 Solution of linear differential equations and simultaneous differential equations using Laplace transforms (Initial value problems)

We have already said that a differential equation with a set of initial conditions is called an initial value problem. However if the boundary conditions are given the problem is called a boundary value problem. Laplace transform serves as a useful tool in solving such problems.

Working procedure for problems

- The given differential equation is expressed in the notation : $y'(t)$, $y''(t)$, $y'''(t)\ldots$ for the derivatives.
- We take Laplace transform on both sides of the given equation.
- We use the expressions for $L[y'(t)]$, $L[y''(t)]\ldots$
- We substitute the given initial conditions and simplify to obtain $L[y(t)]$ as a function of s
- We find the inverse to obtain $y(t)$

WORKED PROBLEMS

74. *Solve by using Laplace transforms :* $\dfrac{d^2y}{dt^2} + k^2 y = 0$ *given that* $y(0) = 2$, $y'(0) = 0$

$s^2y''(t) - sy(0) + y'(0)$

>> The given equation is $y''(t) + k^2 y(t) = 0$

Taking Laplace transform on both sides we have,

$$L[y''(t)] + k^2 L[y(t)] = L(0)$$

ie., $$\left\{s^2 L[y(t)] - s\,y(0) - y'(0)\right\} + k^2 L[y(t)] = 0$$

Using the given initial conditions we obtain,

$$(s^2+k^2)\ L[y(t)]-2s = 0 \quad \text{or} \quad L[y(t)] = \frac{2s}{s^2+k^2}$$

$$\therefore \quad y(t) = 2L^{-1}\left[\frac{s}{s^2+k^2}\right] = 2\cos kt$$

Thus $\boldsymbol{y(t) = 2\cos kt}$

---

76. Solve $y'''+2y''-y'-2y = 0$ given $y(0) = y'(0) = 0$ and $y''(0) = 6$ by using Laplace transform method.

>> Taking Laplace transform on both sides of the given equation,

$$L[y'''(t)]+2L[y''(t)]-L[y'(t)]-2L[y(t)] = L(0)$$

ie., $$\left\{s^3L[y(t)]-s^2.y(0)-sy'(0)-y''(0)\right\}+2\left\{s^2L[y(t)]-sy(0)-y'(0)\right\}$$
$$-\left\{sL[y(t)]-y(0)\right\}-2L[y(t)] = 0$$

Using the given initial conditions we obtain,

$$L[y(t)]\left\{s^3+2s^2-s-2\right\}-6 = 0$$

ie., $$L[y(t)]\left\{s^2(s+2)-1(s+2)\right\} = 6$$

or $$L[y(t)]\left\{(s+2)(s^2-1)\right\} = 6$$

or $$L[y(t)] = \frac{6}{(s+2)(s-1)(s+1)}$$

$$\therefore \quad y(t) = L^{-1}\left\{\frac{6}{(s+2)(s-1)(s+1)}\right\}$$

Let $$\frac{6}{(s+2)(s-1)(s+1)} = \frac{A}{s+2}+\frac{B}{s-1}+\frac{C}{s+1}$$

or $$6 = A(s-1)(s+1)+B(s+2)(s+1)+C(s+2)(s-1)$$

Put $s=-2$: $6 = A(-3)(-1)$ $\quad\therefore A = 2$

Put $s=1$: $6 = B(3)(2)$ $\quad\therefore B = 1$

Put $s=-1$: $6 = C(1)(-2)$ $\quad\therefore C = -3$

Hence $$\frac{6}{(s+2)(s-1)(s+1)} = \frac{2}{s+2}+\frac{1}{s-1}+\frac{-3}{s+1}$$

$$\therefore \quad L^{-1}\left\{\frac{6}{(s+2)(s-1)(s+1)}\right\} = 2L^{-1}\left(\frac{1}{s+2}\right)+L^{-1}\left(\frac{1}{s-1}\right)-3L^{-1}\left(\frac{1}{s+1}\right)$$

Thus $\boldsymbol{y(t) = 2e^{-2t}+e^{t}-3e^{-t}}$

---

76. *Solve* $\dfrac{d^4 y}{dt^4} - k^4 y = 0$ *given* $y(0) = 1$ *and* $y'(0) = 0 = y''(0) = y'''(0)$

>> The given equation is $y^{(4)}(t) - k^4 y(t) = 0$

Taking Laplace transform on both sides we have,

$$L\left[y^{(4)}(t)\right] - k^4 L\left[y(t)\right] = L(0)$$

ie., $\left\{s^4 L[y(t)] - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0)\right\} - k^4 L[y(t)] = 0$

Using the given initial conditions we obtain,

$$L[y(t)]\,[s^4 - k^4] - s^3 \cdot 1 = 0 \text{ or } L[y(t)] = \frac{s^3}{s^4 - k^4}$$

$$\therefore \quad y(t) = L^{-1}\left[\frac{s^3}{s^4 - k^4}\right]$$

Now $s^4 - k^4 = (s^2 - k^2)(s^2 + k^2) = (s-k)(s+k)(s^2+k^2)$

Let $$\frac{s^3}{(s-k)(s+k)(s^2+k^2)} = \frac{A}{(s-k)} + \frac{B}{(s+k)} + \frac{Cs+D}{s^2+k^2}$$

ie., $s^3 = A(s+k)(s^2+k^2) + B(s-k)(s^2+k^2) + (Cs+D)(s-k)(s+k) \ldots (1)$

Put $s = k$: $k^3 = A(2k)(2k^2)$ $\therefore$ $A = 1/4$

Put $s = -k$: $-k^3 = B(-2k)(2k^2)$ $\therefore$ $B = 1/4$

Put $s = 0$: $0 = (1/4)(k)(k^2) + (1/4)(-k)(k^2) + D(-k^2)$

ie., $0 = (k^3/4) - (k^3/4) + D(-k^2)$ $\therefore$ $D = 0$

Comparing the coefficient of s^3 on both sides of (1) we have

$$1 = A + B + C \quad ie., \; 1 = 1/4 + 1/4 + C \quad \therefore \quad C = 1/2$$

Hence $$\frac{s^3}{(s-k)(s+k)(s^2+k^2)} = \frac{1}{4}\frac{1}{s-k} + \frac{1}{4}\frac{1}{s+k} + \frac{1}{2}\frac{s}{s^2+k^2}$$

$$\therefore \quad L^{-1}\left[\frac{s^3}{s^4-k^4}\right] = \frac{1}{4}L^{-1}\left[\frac{1}{s-k}\right] + \frac{1}{4}L^{-1}\left[\frac{1}{s+k}\right] + \frac{1}{2}L^{-1}\left[\frac{s}{s^2+k^2}\right]$$

ie., $y(t) = \frac{1}{4}e^{kt} + \frac{1}{4}e^{-kt} + \frac{1}{2}\cos kt = \frac{1}{4}(e^{kt} + e^{-kt}) + \frac{1}{2}\cos kt$

ie., $y(t) = \frac{1}{4}(2\cos h\, kt) + \frac{1}{2}\cos kt.$

Thus $\mathbf{y(t) = \frac{1}{2}(\cosh kt + \cos kt)}$

---

77. *Solve the following initial value problem by using Laplace transforms :*
$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t};\ y(0) = 0,\ y'(0) = 0$

>> The given equation is $y''(t) + 4y'(t) + 4y(t) = e^{-t}$

Taking Laplace transform on both sides we have,

$$L[y''(t)] + 4L[y'(t)] + 4L[y(t)] = L(e^{-t})$$

ie., $\left\{s^2 L[y(t)] - s\,y(0) - y'(0)\right\} + 4\left\{s\,L[y(t)] - y(0)\right\} + 4L[y(t)] = \frac{1}{s+1}$

Using the given initial conditions we obtain,

$$L[y(t)]\left\{s^2 + 4s + 4\right\} = \frac{1}{s+1} \text{ or } Ly(t) = \frac{1}{(s+1)(s+2)^2}$$

$\therefore$ $y(t) = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$

Let $\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$

Multiplying with $(s+1)(s+2)^2$ we obtain

$$1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

Putting $s = -1$ we get $A = 1$

Putting $s = -2$ we get $C = -1$

Putting $s = 0$ we have $1 = 1(4) + B(2) - 1(1)$ $\therefore$ $B = -1$

Hence $\frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} + \frac{-1}{s+2} + \frac{-1}{(s+2)^2}$

$\therefore$ $L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] - L^{-1}\left[\frac{1}{(s+2)^2}\right]$

ie., $y(t) = e^{-t} - e^{-2t} - e^{-2t} L^{-1}\left(\frac{1}{s^2}\right)$

Thus $\mathbf{y(t) = e^{-t} - e^{-2t} - e^{-2t} t = e^{-t} - (1+t)\, e^{-2t}}$

---

78. *Employ Laplace transform to solve the equation* : $y'' + 5y' + 6y = 5e^{2x}$, $y(0) = 2,\ y'(0) = 1$

>> Taking Laplace transform on both sides of the given equation we have,

$$L[y''(x)] + 5L[y'(x)] + 6L[y(x)] = 5L(e^{2x})$$

ie., $$\{s^2 L[y(x)] - s\,y(0) - y'(0)\} + 5\{s\,L[y(x)] - y(0)\} + 6L[y(x)] = \frac{5}{s-2}$$

Using the given initial conditions we obtain,

$$(s^2 + 5s + 6)\, L[y(x)] - 2s - 1 - 10 = \frac{5}{s-2}$$

ie., $$(s^2 + 5s + 6)\, L[y(x)] = (2s + 11) + \frac{5}{s-2}$$

$$L[y(x)] = \frac{(2s+11)(s-2)+5}{(s-2)(s^2+5s+6)} = \frac{2s^2 + 7s - 17}{(s-2)(s+2)(s+3)}$$

$\therefore$ $$y(x) = L^{-1}\left[\frac{2s^2 + 7s - 17}{(s-2)(s+2)(s+3)}\right]$$

Let $$\frac{2s^2 + 7s - 17}{(s-2)(s+2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s+3}$$

or $$2s^2 + 7s - 17 = A(s+2)(s+3) + B(s-2)(s+3) + C(s-2)(s+2)$$

Put $s = 2$: $5 = A(4)(5)$ $\therefore$ $A = 1/4$

Put $s = -2$: $-23 = B(-4)(1)$ $\therefore$ $B = 23/4$

Put $s = -3$: $-20 = C(-5)(-1)$ $\therefore$ $C = -4$

Hence $$L^{-1}\left[\frac{2s^2 + 7s - 17}{(s-2)(s+2)(s+3)}\right]$$

$$= \frac{1}{4} L^{-1}\left[\frac{1}{s-2}\right] + \frac{23}{4} L^{-1}\left[\frac{1}{s+2}\right] - 4L^{-1}\left[\frac{1}{s+3}\right]$$

Thus $\mathbf{y(x) = \frac{1}{4} e^{2x} + \frac{23}{4} e^{-2x} - 4\,e^{-3x}}$

---

79. *Using Laplace transform technique solve $x'' - 2x' + x = e^{2t}$ with $x(0) = 0,\ x'(0) = -1$*

>> Taking Laplace transform on both sides of the given equation we have,

$$L[x''(t)] - 2L[x'(t)] + L[x(t)] = L(e^{2t})$$

ie., $$\{s^2 L[x(t)] - s x(0) - x'(0)\} - 2\{s L[x(t)] - x(0)\} + Lx(t) = \frac{1}{s-2}$$

Using the given initial conditions we obtain,

$$\{s^2 - 2s + 1\} L[x(t)] + 1 = \frac{1}{s-2}$$

ie., $$(s-1)^2 L[x(t)] = \frac{1}{s-2} - 1 = \frac{3-s}{(s-2)}$$

or $$L[x(t)] = \frac{3-s}{(s-1)^2(s-2)}$$

$\therefore$ $$x(t) = L^{-1}\left[\frac{3-s}{(s-1)^2(s-2)}\right]$$

Let $$\frac{3-s}{(s-1)^2(s-2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-2}$$

Multiplying by $(s-1)^2(s-2)$ we get,

$$3 - s = A(s-1)(s-2) + B(s-2) + C(s-1)^2 \quad \ldots (1)$$

Put $s = 1$: $2 = B(-1)$ $\therefore$ $B = -2$

Put $s = 2$: $1 = C(1)$ $\therefore$ $C = 1$

Equating the coefficient of s^2 on both sides of (1) we get,

$0 = A + C$. $\therefore$ $A = -1$

Hence $$L^{-1}\left[\frac{3-s}{(s-1)^2(s-2)}\right] = -L^{-1}\left[\frac{1}{s-1}\right] - 2L^{-1}\left[\frac{1}{(s-1)^2}\right] + L^{-1}\left[\frac{1}{s-2}\right]$$

Thus $$\mathbf{x(t) = -e^t - 2e^t \cdot t + e^{2t} = e^{2t} - (1+2t)e^t}$$

---

80. *Solve the D.E. $y'' + 4y' + 3y = e^{-t}$ with $y(0) = 1 = y'(0)$ using Laplace transforms.*

>> Taking Laplace transform on both sides of the given equation we have,

$$L[y''(t)] + 4L[y'(t)] + 3L[y(t)] = L(e^{-t})$$

ie., $$\{s^2 L[y(t)] - s\,y(0) - y'(0)\} + 4\{s L[y(t)] - y(0)\} + 3L[y(t)] = \frac{1}{s+1}$$

Using the given initial conditions we obtain,

$$(s^2+4s+3)\,L[y(t)]-s-1-4 = \frac{1}{s+1}$$

ie., $$(s^2+4s+3)\,L[y(t)] = (s+5)+\frac{1}{(s+1)}$$

ie., $$(s+1)(s+3)\,L[y(t)] = \frac{s^2+6s+6}{s+1}$$

or $$L[y(t)] = \frac{s^2+6s+6}{(s+1)^2(s+3)}$$

$\therefore$ $$y(t) = L^{-1}\left[\frac{s^2+6s+6}{(s+1)^2(s+3)}\right]$$

Let $$\frac{s^2+6s+6}{(s+1)^2(s+3)} = \frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{s+3}$$

Multiplying by $(s+1)^2(s+3)$ we get,

$$s^2+6s+6 = A(s+1)(s+3)+B(s+3)+C(s+1)^2 \qquad \ldots(1)$$

Put $s=-1$: $1 = B(2)$ $\quad\therefore\ B = 1/2$

Put $s=-3$: $-3 = C(4)$ $\quad\therefore\ C = -3/4$

Equating the coefficient of s^2 on both sides of (1) we get,

$$1 = A+C \quad \therefore\ A = 7/4$$

Hence $$L^{-1}\left[\frac{s^2+6s+6}{(s+1)^2(s+3)}\right] = \frac{7}{4}L^{-1}\left[\frac{1}{s+1}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}\right]-\frac{3}{4}L^{-1}\left[\frac{1}{s+3}\right]$$

Thus $$y(t) = \frac{7}{4}e^{-t}+\frac{1}{2}e^{-t}\cdot t-\frac{3}{4}e^{-3t}$$

---

81. *Solve by using Laplace transforms* $\dfrac{d^2x}{dt^2}+2\dfrac{dx}{dt}+x = 3te^{-t}$ *given that* $x = 4$, $\dfrac{dx}{dt} = 2$ *when* $t = 0$.

>> The given equation is $x''(t)+2x'(t)+x(t) = 3te^{-t}$.

Initial conditions are $x(0) = 4$, $x'(0) = 2$

Taking Laplace transform on both sides of the equation we have,

$$L[x''(t)]+2L[x'(t)]+L[x(t)] = 3L(e^{-t}\cdot t)$$

ie., $\left\{s^2 L[x(t)] - s\,x(0) - x'(0)\right\} + 2\left\{s L[x(t)] - x(0)\right\} + L[x(t)] = \dfrac{3}{(s+1)^2}$

Using the given initial conditions we obtain,

$$(s^2+2s+1)\,L[x(t)] - 4s - 2 - 8 = \frac{3}{(s+1)^2}$$

ie., $$(s+1)^2\,L[x(t)] = (4s+10) + \frac{3}{(s+1)^2}$$

or $$L[x(t)] = \frac{4s+10}{(s+1)^2} + \frac{3}{(s+1)^4}$$

$\therefore$ $$x(t) = L^{-1}\left[\frac{4(s+1)+6}{(s+1)^2}\right] + L^{-1}\left[\frac{3}{(s+1)^4}\right]$$

$$= e^{-t}L^{-1}\left[\frac{4s+6}{s^2}\right] + 3e^{-t}L^{-1}\left[\frac{1}{s^4}\right]$$

$$= e^{-t}\left\{4L^{-1}\left(\frac{1}{s}\right) + 6L^{-1}\left(\frac{1}{s^2}\right) + 3L^{-1}\left(\frac{1}{s^4}\right)\right\}$$

$$= e^{-t}(4 + 6t + 3\cdot t^3/6)$$

Thus $\boldsymbol{x(t) = e^{-t}(4 + 6t + t^3/2)}$

---

82. *Solve by using Laplace transforms* $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 2y = 5\sin t$ *given that* $y(0) = 0 = y'(0)$

>> The given equation is $y''(t) + 2y'(t) + 2y(t) = 5\sin t$

Taking Laplace transform on both sides we have,

$$L[y''(t)] + 2L[y'(t)] + 2L[y(t)] = 5L(\sin t).$$

ie., $\left\{s^2 L[y(t)] - s\,y(0) - y'(0)\right\} + 2\left\{sL[y(t)] - y(0)\right\} + 2L[y(t)] = \dfrac{5}{s^2+1}$

Using the given initial conditions we obtain,

$$L[y(t)]\left\{s^2+2s+2\right\} = \frac{5}{s^2+1} \quad \text{or} \quad L[y(t)] = \frac{5}{(s^2+1)(s^2+2s+2)}$$

$\therefore$ $$y(t) = L^{-1}\left[\frac{5}{(s^2+1)(s^2+2s+2)}\right]$$

Let $$\frac{5}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}$$

ie., $5 = (As + B)(s^2 + 2s + 2) + (Cs + D)(s^2 + 1)$

ie., $5 = (A + C)s^3 + (2A + B + D)s^2 + (2A + 2B + C)s + (2B + D)$

Comparing the coefficients on both sides, we get

$$A + C = 0 \; ; \; 2A + B + D = 0 \; ; \; 2A + 2B + C = 0 \; ; \; 2B + D = 5$$

Solving these simultaneously we obtain

$$A = -2, \; B = 1, \; C = 2, \; D = 3$$

Hence $\dfrac{5}{(s^2+1)(s^2+2s+2)} = \dfrac{-2s+1}{s^2+1} + \dfrac{2s+3}{s^2+2s+2}$

$$\therefore \quad L^{-1}\left[\frac{5}{(s^2+1)(s^2+2s+2)}\right]$$

$$= -2L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right) + L^{-1}\left(\frac{2s+3}{s^2+2s+2}\right)$$

ie., $y(t) = -2\cos t + \sin t + L^{-1}\left\{\dfrac{2(s+1)+1}{(s+1)^2+1}\right\}$

$$= -2\cos t + \sin t + e^{-t}L^{-1}\left\{\frac{2s+1}{s^2+1}\right\}$$

$$= -2\cos t + \sin t + e^{-t}\left[2L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

Thus $\mathbf{y(t) = -2\cos t + \sin t + e^{-t}(2\cos t + \sin t)}$

---

83. *Solve :* $y'' + 6y' + 9y = 12t^2e^{-3t}$ *subject to the conditions,* $y(0) = 0 = y'(0)$ *by using Laplace transforms.*

>> The given equation is $y''(t) + 6y'(t) + 9y(t) = 12t^2e^{-3t}$

Taking Laplace transform on both sides we have,

$$L[y''(t)] + 6L[y'(t)] + 9L[y(t)] = 12L[e^{-3t}t^2]$$

ie., $\{s^2L[y(t)] - sy(0) - y'(0)\} + 6\{sL[y(t)] - y(0)\} + 9L[y(t)] = \dfrac{12 \cdot 2!}{(s+3)^3}$

Using the given initial conditions we obtain,

$$(s^2 + 6s + 9)\,L[y(t)] = \frac{24}{(s+3)^3} \quad \text{or} \quad L[y(t)] = \frac{24}{(s+3)^5}$$

$\therefore \quad y(t) = L^{-1}\left[\dfrac{24}{(s+3)^5}\right]$

$$y(t) = 24e^{-3t}L^{-1}\left(\frac{1}{s^5}\right) = 24e^{-3t}\frac{t^4}{4!}$$

Thus $\quad y(t) = e^{-3t}t^4$

---

84. *Solve by using Laplace transform method $y''(t)+y(t) = H(t-1)$ given $y(0) = 0$ and $y'(0) = 1$*

>> Taking Laplace transform on both sides of the given equation we have,

$$L[y''(t)]+L[y(t)] = L[H(t-1)]$$

ie., $\quad \left\{s^2 L[y(t)]-sy(0)-y'(0)\right\}+L[y(t)] = \dfrac{e^{-s}}{s}$

Using the given initial conditions we obtain,

$$(s^2+1)\,L[y(t)]-1 = \frac{e^{-s}}{s} \quad \text{or} \quad (s^2+1)\,L[y(t)] = 1+\frac{e^{-s}}{s}$$

$\therefore \quad L[y(t)] = \dfrac{1}{s^2+1}+\dfrac{e^{-s}}{s(s^2+1)}$

$\Rightarrow \quad y(t) = L^{-1}\left[\dfrac{1}{s^2+1}\right]+L^{-1}\left[\dfrac{e^{-s}}{s(s^2+1)}\right]$

ie., $\quad y(t) = \sin t+L^{-1}\left[\dfrac{e^{-s}}{s(s^2+1)}\right] \qquad \ldots(1)$

In respect of the second term, let $\overline{f}(s) = \dfrac{1}{s(s^2+1)}$

let $\quad \overline{f}(s) = \dfrac{1}{s(s^2+1)} = \dfrac{1}{s}-\dfrac{s}{s^2+1}$ by partial fractions.

Now, $\quad L^{-1}[\overline{f}(s)] = L^{-1}\left[\dfrac{1}{s}\right]-L^{-1}\left[\dfrac{s}{s^2+1}\right]$

ie., $\quad f(t) = 1-\cos t$

We also have, $L^{-1}[e^{-s}\,\overline{f}(s)] = f(t-1)H(t-1)$

ie., $$L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right] = [1-\cos(t-1)]H(t-1) \quad \ldots(2)$$

Thus by using (2) in (1) we get,

$$y(t) = \sin t + [1-\cos(t-1)]H(t-1)$$

---

85. *Solve by using Laplace transforms :* $y''(t)+y(t) = F(t)$ *where* $F(t) = \begin{cases} 4, & 0<t<1 \\ 3, & t>1 \end{cases}$ *subject to the conditions,* $y(0) = 0$ *and* $y'(0) = 1$

>> Taking Laplace transform on both sides of the given equation we have,

$$L[y''(t)]+L[y(t)] = L[F(t)]$$

ie., $$\{s^2 L[y(t)]-sy(0)-y'(0)\}+L[y(t)] = L[F(t)]$$

Using the given initial conditions we obtain,

$$(s^2+1)\,L[y(t)]-1 = L[F(t)] \quad \ldots(1)$$

Now $$L[F(t)] = \int_0^\infty e^{-st}F(t)\,dt = \int_0^1 e^{-st}F(t)\,dt+\int_1^\infty e^{-st}F(t)\,dt$$

ie., $$L[F(t)] = \int_0^1 4e^{-st}\,dt+\int_1^\infty 3e^{-st}\,dt$$

$$= 4\left[\frac{e^{-st}}{-s}\right]_0^1+3\left[\frac{e^{-st}}{-s}\right]_1^\infty = \frac{-4}{s}(e^{-s}-1)-\frac{3}{s}(0-e^{-s})$$

Hence, $$L[F(t)] = \frac{4}{s}-\frac{1}{s}e^{-s} \quad \ldots(2)$$

Using (2) in the R.H.S of (1) we get,

$$(s^2+1)\,Ly(t) = 1+\frac{4}{s}-\frac{e^{-s}}{s}$$

or $$Ly(t) = \frac{1}{s^2+1}+\frac{4}{s(s^2+1)}-\frac{e^{-s}}{s(s^2+1)}$$

$\therefore$ $$y(t) = L^{-1}\left[\frac{1}{s^2+1}\right]+4L^{-1}\left[\frac{1}{s(s^2+1)}\right]-L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right]$$

[*Refer Problem-84 for the inverse of the second and third terms*]

Thus $$y(t) = \sin t+4(1-\cos t)-[1-\cos(t-1)]H(t-1)$$

---

86. *Solve the following boundary value problem by using Laplace transforms:* $y''(t)+y(t)=0$; $y(0)=2,\ y(\pi/2)=1$

>> Taking Laplace transform on both sides of the given equation we have,

$$L[y''(t)]+Ly(t)=L(0)$$

ie., $\{s^2 Ly(t)-s\,y(0)-y'(0)\}+Ly(t)=0$ ...(1)

Let us assume $y'(0)=a$ where a is a constant to be found later and we have $y(0)=2$ by data.

Hence (1) becomes,

$$(s^2+1)\,L[y(t)]-2s-a=0$$

ie., $(s^2+1)\,L[y(t)]=2s+a$ or $L[y(t)]=\dfrac{2s+a}{s^2+1}$

$\therefore$ $y(t)=2L^{-1}\left[\dfrac{s}{s^2+1}\right]+a\,L^{-1}\left[\dfrac{1}{s^2+1}\right]$

ie., $y(t)=2\cos t+a\sin t$ ...(2)

Now we shall use the condition $y(\pi/2)=1$

Hence (2) becomes $y(\pi/2)=2\cos(\pi/2)+a\sin(\pi/2)$

ie., $1=0+a \quad \therefore \quad a=1$

Thus $\boldsymbol{y(t)=2\cos t+\sin t}$

---

87. *Solve by using Laplace transforms :* $\dfrac{dy}{dt}+2y+\int_0^t y\,dt=t\,e^{-t}$ *given that* $y(0)=1$

>> The given equation is $y'(t)+2y(t)+\int_0^t y\,dt=t\,e^{-t}$

Taking Laplace transform on both sides we get,

$$L[y'(t)]+2L[y(t)]+L\left[\int_0^t y(t)\,dt\right]=L[e^{-t}t]$$

ie., $\{s\,L[y(t)]-y(0)\}+2\,Ly(t)+\dfrac{L[y(t)]}{s}=\dfrac{1}{(s+1)^2}$

ie., $\left(s+2+\frac{1}{s}\right) L[y(t)] = 1+\frac{1}{(s+1)^2}$, by using $y(0)=1$

ie., $\left[\frac{s^2+2s+1}{s}\right] L[y(t)] = 1+\frac{1}{(s+1)^2}$

or $L[y(t)] = \frac{s}{(s+1)^2}\left[1+\frac{1}{(s+1)^2}\right] = \frac{s}{(s+1)^2}+\frac{s}{(s+1)^4}$

$\therefore \quad y(t) = L^{-1}\left[\frac{s}{(s+1)^2}\right]+L^{-1}\left[\frac{s}{(s+1)^4}\right]$

$= L^{-1}\left[\frac{(s+1)-1}{(s+1)^2}\right]+L^{-1}\left[\frac{(s+1)-1}{(s+1)^4}\right]$

$= e^{-t}L^{-1}\left[\frac{s-1}{s^2}\right]+e^{-t}L^{-1}\left[\frac{s-1}{s^4}\right]$

$= e^{-t}\left\{L^{-1}\left(\frac{1}{s}\right)-L^{-1}\left(\frac{1}{s^2}\right)+L^{-1}\left(\frac{1}{s^3}\right)-L^{-1}\left(\frac{1}{s^4}\right)\right\}$

Thus $y(t) = e^{-t}(1-t+t^2/2-t^3/6)$

---

88. *Find $f(t)$ from the following equation : $f'(t) = t+\int_0^t f(t-u)\cos u\, du$; $f(0)=4$*

>> Taking Laplace transform on both sides of the equation we have,

$$L[f'(t)] = L(t)+L\left[\int_0^t f(t-u)\cos u\, du\right]$$

ie., $sL[f(t)]-f(0) = \frac{1}{s^2}+L\left[\int_0^t f(t-u)\cos u\, du\right]$

ie., $s\bar{f}(s) = 4+\frac{1}{s^2}+L\left[\int_0^t f(t-u)\cos u\, du\right] \qquad \ldots(1)$

by using $f(0)=4$

We have convolution theorem in the form:

$$L\left[\int_0^t f(t-u)\,g(u)\,du\right] = \overline{f}(s)\cdot\overline{g}(s)$$

Taking $g(u) = \cos u,\ \overline{g}(s) = L[g(u)] = \dfrac{s}{s^2+1}$

$$\therefore \quad L\left[\int_0^t f(t-u)\cos u\,du\right] = \overline{f}(s)\cdot\frac{s}{s^2+1} \qquad \ldots(2)$$

Using (2) in the R.H.S of (1) we get,

$$s\overline{f}(s) = 4+\frac{1}{s^2}+\overline{f}(s)\cdot\frac{s}{s^2+1}$$

ie., $\overline{f}(s)\left[\dfrac{s^3}{s^2+1}\right] = \dfrac{4s^2+1}{s^2}$ or $\overline{f}(s) = \dfrac{(4s^2+1)(s^2+1)}{s^5}$

ie., $\overline{f}(s) = \dfrac{4s^4+5s^2+1}{s^5}$ or $\overline{f}(s) = \dfrac{4}{s}+\dfrac{5}{s^3}+\dfrac{1}{s^5}$

Now $L^{-1}[\overline{f}(s)] = 4L^{-1}\left(\dfrac{1}{s}\right)+5L^{-1}\left(\dfrac{1}{s^3}\right)+L^{-1}\left(\dfrac{1}{s^5}\right)$

ie., $f(t) = 4\cdot 1+5\cdot\dfrac{t^2}{2!}+\dfrac{t^4}{4!}$

Thus $\boldsymbol{f(t) = 4+\dfrac{5t^2}{2}+\dfrac{t^4}{24}}$

---

89. ***Solve by using Laplace transforms :***

$$\frac{d^2x}{dt^2}+\omega^2 x = a\sin(\omega t+\alpha),\ x(0) = 0 = x'(0)$$

>> The given equation is $x''(t)+\omega^2 x(t) = a\sin(\omega t+\alpha)$

ie., $x''(t)+\omega^2 x(t) = a\{\sin\omega t\cos\alpha+\cos\omega t\sin\alpha\}$

Taking Laplace transform on both sides we have,

$$L[x''(t)]+\omega^2 Lx(t) = a\cos\alpha\,L(\sin\omega t)+a\sin\alpha\,L(\cos\omega t)$$

ie., $\{s^2Lx(t)-sx(0)-x'(0)\}+\omega^2 Lx(t) = a\cos\alpha\cdot\dfrac{\omega}{s^2+\omega^2}+a\sin\alpha\cdot\dfrac{s}{s^2+\omega^2}$

ie., $(s^2+\omega^2)\,Lx(t) = \dfrac{a\cos\alpha\cdot\omega}{s^2+\omega^2}+\dfrac{a\sin\alpha\cdot s}{s^2+\omega^2}$

or $Lx(t) = \dfrac{a\cos\alpha\cdot\omega}{(s^2+\omega^2)^2}+a\sin\alpha\cdot\dfrac{s}{(s^2+\omega^2)^2}$

$\therefore$ $x(t) = \omega a\cos\alpha\, L^{-1}\left[\dfrac{1}{(s^2+\omega^2)^2}\right]+a\sin\alpha\, L^{-1}\left[\dfrac{s}{(s^2+\omega^2)^2}\right]$

Recollecting the standard inverse Laplace transforms we have

$$x(t) = \omega a\cos\alpha\cdot\frac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)+a\sin\alpha\cdot\frac{t\sin\omega t}{2\omega}$$

$$= \frac{a\cos\alpha}{2\omega^2}\sin\omega t-\frac{a\cos\alpha}{2\omega}t\cos\omega t+\frac{a\sin\alpha}{2\omega}t\sin\omega t$$

$$= \frac{a\cos\alpha}{2\omega^2}\sin\omega t-\frac{at}{2\omega}(\cos\omega t\cos\alpha-\sin\omega t\sin\alpha)$$

$$= \frac{a\cos\alpha}{2\omega^2}\sin\omega t-\frac{at}{2\omega}\cos(\omega t+\alpha)$$

Thus $x(t) = \dfrac{a}{2\omega^2}\{\cos\alpha\sin\omega t-\omega t\cos(\omega t+\alpha)\}$

Solution of simultaneous differential equations

90. *Given that* $\dfrac{dx}{dt}+4y = 0$, $\dfrac{dy}{dt}-9x = 0$ *and* $x(0) = 2$, $y(0) = 1$, *use Laplace transform method to find* x *and* y *in terms of* t.

\>\> The given equations are $x'(t)+4y(t) = 0$, $-9x(t)+y'(t) = 0$,

Taking Laplace transform on both sides of these equations we have,

$$L[x'(t)]+4L[y(t)] = L(0)$$

$$-9L[x(t)]+L[y'(t)] = L(0)$$

ie., $sL[x(t)]-x(0)+4L[y(t)] = 0$

$$-9L[x(t)]+sL[y(t)]-y(0) = 0$$

Using the given initial conditions we obtain,

$$sL[x(t)]+4L[y(t)] = 2$$

$$-9L[x(t)]+sL[y(t)] = 1$$

or $9s\,L[x(t)]+36L[y(t)] = 18$

$$-9s\,L[x(t)]+s^2L[y(t)] = s$$

Adding, $(s^2+36)\,L[y(t)] = s+18$

or $\quad L[y(t)] = \dfrac{s+18}{s^2+36}$

$\therefore \quad y(t) = L^{-1}\left(\dfrac{s}{s^2+6^2}\right) + 3L^{-1}\left(\dfrac{6}{s^2+6^2}\right)$

Thus $\quad \mathbf{y(t) = \cos 6t + 3\sin 6t}$ ... (1)

Consider $\quad \dfrac{dy}{dt} - 9x = 0$ or $x = \dfrac{1}{9}\dfrac{dy}{dt}$

ie., $\quad x(t) = \dfrac{1}{9}(-6\sin 6t + 18\cos 6t)$

Thus $\mathbf{x(t) = \dfrac{2}{3}(3\cos 6t - \sin 6t)}$ ... (2)

(1) and (2) represents the solution of the given equations.

---

91. ***Solve by using Laplace transforms*** $\dfrac{dx}{dt} - 2y = \cos 2t,\ \dfrac{dy}{dt} + 2x = \sin 2t$; $x = 1,\ y = 0$ *at* $t = 0$

>> The given system of equations are,

$x'(t) - 2y(t) = \cos 2t$ ... (1)

$2x(t) + y'(t) = \sin 2t$ ... (2)

with the initial conditions $x(0) = 1$ and $y(0) = 0$
Taking Laplace transform on both sides of (1) and (2) we have

$L[x'(t)] - 2L[y(t)] = L(\cos 2t)$

$2L[x(t)] + L[y'(t)] = L(\sin 2t)$

ie., $\quad \{sL[x(t)] - x(0)\} - 2L[y(t)] = s/s^2+4$

$2L[x(t)] + \{sL[y(t)] - y(0)\} = 2/s^2+4$

Using the given initial conditions we have,

$sL[x(t)] - 2L[y(t)] = 1 + (s/s^2+4)$

$2L[x(t)] + sL[y(t)] = 2/s^2+4$

Let us multiply (3) by s and (4) by 2

$s^2 L[x(t)] - 2sL[y(t)] = s + (s^2/s^2+4)$

$4L[x(t)] + 2sL[y(t)] = 4/s^2+4$

Adding we get, $(s^2+4)\ L[x(t)] = s+\dfrac{s^2}{s^2+4}+\dfrac{4}{s^2+4}$

ie., $(s^2+4)\ L[x(t)] = s+1$

or $L[x(t)] = \dfrac{s+1}{s^2+4}$

$\therefore \quad x(t) = L^{-1}\left(\dfrac{s}{s^2+2^2}\right)+L^{-1}\left(\dfrac{1}{s^2+2^2}\right)$

Thus $\mathbf{x(t) = \cos 2t + \frac{1}{2}\sin 2t}$ ... (5)

To find $y(t)$, let us consider $\dfrac{dx}{dt}-2y = \cos 2t$

$\therefore \quad y = \dfrac{1}{2}\left[\dfrac{dx}{dt}-\cos 2t\right] = \dfrac{1}{2}\left[\dfrac{d}{dt}\left(\cos 2t+\dfrac{1}{2}\sin 2t\right)-\cos 2t\right]$

ie., $y(t) = \dfrac{1}{2}[-2\sin 2t+\cos 2t-\cos 2t] = -\sin 2t$

Thus $\mathbf{y(t) = -\sin 2t}$ ... (6)

(5) and (6) represents the required solution.

---

92. *Solve the following system of equations using Laplace transforms* $\dfrac{dx}{dt}-y = e^t,\ \dfrac{dy}{dt}+x = \sin t$; *given that* $x=1,\ y=0$ *at* $t=0$

>> The given equations are,

$x'(t)-y(t) = e^t$ ... (1)

$x(t)+y'(t) = \sin t$ ... (2)

$x(0) = 1$ and $y(0) = 0$ are the given initial conditions.

Taking Laplace transform on bothsides of (1) and (2) we have,

$L[x'(t)]-L[y(t)] = L(e^t)$

$L[x(t)]+L[y'(t)] = L(\sin t)$

ie., $s\,L[x(t)]-x(0)-L[y(t)] = 1/s-1$

$L[x(t)]+s\,L[y(t)]-y(0) = 1/s^2+1$

Using the given initial conditions we obtain,

$s\,L[x(t)]-L[y(t)] = 1+(1/s-1)$

$L[x(t)]+s\,L[y(t)] = 1/s^2+1$

or $\quad s^2 L[x(t)] - s L[y(t)] = s + (s/s-1)$

$$L[x(t)] \quad + s L[y(t)] = 1/s^2 + 1$$

Adding we get, $(s^2+1)\, L[x(t)] = s + \dfrac{s}{s-1} + \dfrac{1}{s^2+1}$

or $\quad L[x(t)] = \dfrac{s}{s^2+1} + \dfrac{s}{(s-1)(s^2+1)} + \dfrac{1}{(s^2+1)^2}$

$$\therefore \quad x(t) = L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{s}{(s-1)(s^2+1)}\right] + L^{-1}\left[\frac{1}{(s^2+1)^2}\right] \qquad \ldots (3)$$

Let $\quad \dfrac{s}{(s-1)(s^2+1)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+1}$

or $\quad s = A(s^2+1) + (Bs+C)(s-1)$

Put $s = 1$: $1 = A(2)$ $\quad \therefore \quad A = 1/2$

Put $s = 0$: $0 = 1/2 + C(-1)$ $\quad \therefore \quad C = 1/2$

Equating the coefficient of s^2 on both sides we get,

$$0 = A + B \quad \therefore \quad B = -1/2$$

Now, $\quad L^{-1}\left[\dfrac{s}{(s-1)(s^2+1)}\right] = \dfrac{1}{2}L^{-1}\left[\dfrac{1}{s-1}\right] - \dfrac{1}{2}L^{-1}\left[\dfrac{s}{s^2+1}\right] + \dfrac{1}{2}L^{-1}\left[\dfrac{1}{s^2+1}\right]$

ie., $\quad L^{-1}\left[\dfrac{s}{(s-1)(s^2+1)}\right] = \dfrac{1}{2}(e^t - \cos t + \sin t) \qquad \ldots (4)$

Further we have $L^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right] = \dfrac{1}{2a^3}(\sin at - a t \cos at)$

$$\therefore \quad L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = \frac{1}{2}(\sin t - t\cos t) \qquad \ldots (5)$$

Hence (3) as a consequence of (4) and (5) becomes,

$$x(t) = \cos t + \frac{1}{2}(e^t - \cos t + \sin t) + \frac{1}{2}(\sin t - t\cos t)$$

Thus $\quad \boldsymbol{x(t) = \dfrac{1}{2}(e^t + \cos t + 2\sin t - t\cos t)} \qquad \ldots (6)$

Also from (1), $y(t) = \dfrac{dx}{dt} - e^t$

ie., $y(t) = \frac{1}{2}(e^t - \sin t + 2\cos t + t\sin t - \cos t) - e^t$

Thus $\mathbf{y(t) = \frac{1}{2}(t\sin t + \cos t - \sin t - e^t)}$ ... (7)

(6) and (7) represents the solution of the given equations.

---

93. *Solve by using Laplace transforms* $\frac{dx}{dt} = 2x - 3y$, $\frac{dy}{dt} = y - 2x$ *given that* $x(0) = 8$ *and* $y(0) = 3$

>> The given equations are, $x'(t) - 2x(t) + 3y(t) = 0$

$$2x(t) + y'(t) - y(t) = 0$$

Taking Laplace transform on both sides of these equations we have,

$$L[x'(t)] - 2L[x(t)] + 3L[y(t)] = 0$$

$$2L[x(t)] + L[y'(t)] - L[y(t)] = 0$$

ie., $sL[x(t)] - x(0) - 2L[x(t)] + 3L[y(t)] = 0$

$$2L[x(t)] + sL[y(t)] - y(0) - L[y(t)] = 0$$

Using the given initial conditions we obtain,

$$(s-2)L[x(t)] + 3L[y(t)] = 8$$

$$2L[x(t)] + (s-1)L[y(t)] = 3$$

or $(s-1)(s-2)L[x(t)] + 3(s-1)L[y(t)] = 8s - 8$

$$6L[x(t)] + 3(s-1)L[y(t)] = 9$$

Subtracting we get, $(s^2 - 3s - 4)L[x(t)] = 8s - 17$

$\therefore$ $x(t) = L^{-1}\left[\frac{8s-17}{s^2-3s-4}\right] = L^{-1}\left[\frac{8s-17}{(s-4)(s+1)}\right]$

Let $\frac{8s-17}{(s-4)(s+1)} = \frac{A}{s-4} + \frac{B}{s+1}$

or $8s - 17 = A(s+1) + B(s-4)$

Put $s = 4$: $15 = 5A$ $\therefore$ $A = 3$

Put $s = -1$: $-25 = -5B$ $\therefore$ $B = 5$

Now $x(t) = 3L^{-1}\left[\frac{1}{s-4}\right] + 5L^{-1}\left[\frac{1}{s+1}\right]$

Thus $\mathbf{x(t) = 3e^{4t} + 5e^{-t}}$ ... (1)

Consider $\dfrac{dx}{dt} = 2x - 3y \quad \therefore \quad y = \dfrac{1}{3}\left[2x - \dfrac{dx}{dt}\right]$

ie., $\quad y(t) = \dfrac{1}{3}\left[2(3e^{4t} + 5e^{-t}) - (12e^{4t} - 5e^{-t})\right] = \dfrac{1}{3}(-6e^{4t} + 15e^{-t})$

Thus $\quad \boldsymbol{y(t) = 5e^{-t} - 2e^{4t}}$ $\quad \ldots (2)$

(1) and (2) represents the solution of the given equations.

---

94. *Solve the following system of equations by using Laplace transforms.* $\dfrac{dx}{dt} + \dfrac{dy}{dt} = 2z, \; \dfrac{dy}{dt} + \dfrac{dz}{dt} = 2x, \; \dfrac{dz}{dt} + \dfrac{dx}{dt} = 2y$ *given that* $x = y = z = 1$ *when* $t = 0$.

\>> The given equaitons are,

$$\left.\begin{aligned} x'(t) + y'(t) &= 2z(t) \\ y'(t) + z'(t) &= 2x(t) \\ z'(t) + x'(t) &= 2y(t) \end{aligned}\right\} \; x(0) = 1 = y(0) = z(0) \text{ are the initial conditions.}$$

Taking Laplace transform on bothsides of these equations we have,

$$sL[x(t)] - x(0) + sL[y(t)] - y(0) = 2L[z(t)]$$
$$sL[y(t)] - y(0) + sL[z(t)] - z(0) = 2L[x(t)]$$
$$sL[z(t)] - z(0) + sL[x(t)] - x(0) = 2L[y(t)]$$

Using the given initial conditions we obtain,

$$sL[x(t)] + sL[y(t)] - 2L[z(t)] = 2 \quad \ldots (1)$$
$$-2L[x(t)] + sL[y(t)] + sL[z(t)] = 2 \quad \ldots (2)$$
$$sL[x(t)] - 2L[y(t)] + sL[z(t)] = 2 \quad \ldots (3)$$

$(1) \times 2 + (2) \times s \qquad (s^2 + 2s)\, L[y(t)] + (s^2 - 4)\, L[z(t)] = 2s + 4 \quad \ldots (4)$

$(1) - (3) \qquad (s+2)\, L[y(t)] - (s+2)\, L[z(t)] = 0 \quad \ldots (5)$

Now $(4) + (s-2) \times (5)$ will give us

$$(s^2 + 2s + s^2 - 4)\, L[y(t)] = 2s + 4$$

ie., $\quad (2s^2 + 2s - 4)\, L[y(t)] = 2(s+2)$

ie., $\quad 2(s^2 + s - 2)\, L[y(t)] = 2(s+2)$

ie., $\quad (s+2)(s-1)\, L[y(t)] = (s+2)$

or $\quad L[y(t)] = \dfrac{1}{s-1} \quad \Rightarrow \quad y(t) = L^{-1}\left[\dfrac{1}{s-1}\right] = e^{t}$

Now (5) becomes $(s+2)\frac{1}{s-1}-(s+2)\,L[z(t)] = 0$

$\therefore \quad L[z(t)] = \frac{1}{s-1} \Rightarrow z(t) = e^t$

But $\quad y'(t)+z'(t) = 2x(t)$

ie., $\quad e^t + e^t = 2x(t)$ or $2x(t) = 2e^t \therefore x(t) = e^t$

Thus $x(t) = e^t = y(t) = z(t)$ represents the solution of the given system of equations.

ADDITIONAL PROBLEMS

95. *Find the inverse Laplace transforms of (i)* $\frac{3s+7}{s^2-2s-3}$ *(ii)* $\log\frac{s^2+1}{s(s+1)}$

>> (i) $s^2-2s-3 = (s-3)(s+1)$

Let $\quad \frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3}+\frac{B}{s+1}$

or $\quad 3s+7 = A(s+1)+B(s-3)$

Put $\quad s = 3 \;:\; 16 = A(4) \quad \therefore A = 4$

Put $\quad s = -1 \;:\; 4 = B(-4) \quad \therefore B = -1$

Hence $L^{-1}\left[\frac{3s+7}{(s-3)(s+1)}\right] = 4L^{-1}\left[\frac{1}{s-3}\right] - L^{-1}\left[\frac{1}{s+1}\right]$

Thus $\quad \mathbf{L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right] = 4e^{3t}-e^{-t}}$

(ii) Let $\bar{f}(s) = \log\frac{s^2+1}{s(s+1)}$: To find $L^{-1}[\bar{f}(s)] = f(t)$.

$\bar{f}(s) = \log(s^2+1)-\log s-\log(s+1)$

$\therefore \quad -\bar{f}'(s) = -\left[\frac{2s}{s^2+1}-\frac{1}{s}-\frac{1}{s+1}\right]$

Now $L^{-1}[-\bar{f}'(s)] = L^{-1}\left[\frac{1}{s}\right]+L^{-1}\left[\frac{1}{s+1}\right]-2L^{-1}\left[\frac{s}{s^2+1}\right]$

ie., $\quad t f(t) = 1+e^{-t}-2\cos t$

Thus $\quad \mathbf{f(t) = \frac{1+e^{-t}-2\cos t}{t}}$

---

96. *Find the inverse Laplace transform of* $\dfrac{1}{s^2(s+1)}$

>> Let $\dfrac{1}{s^2(s+1)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s+1}$

or $1 = As(s+1) + B(s+1) + Cs^2$

Put $s = 0 \quad : \quad 1 = B$

Put $s = -1 \quad : \quad 1 = C$

Equating the coefficient of s^2 on both sides we get

$0 = A + C$ and hence $A = -1$

Now $L^{-1}\left[\dfrac{1}{s^2(s+1)}\right] = -L^{-1}\left[\dfrac{1}{s}\right] + L^{-1}\left[\dfrac{1}{s^2}\right] + L^{-1}\left[\dfrac{1}{s+1}\right]$

Thus $L^{-1}\left[\dfrac{1}{s^2(s+1)}\right] = -1 + t + e^{-t}$

---

97. *Find the inverse Laplace transform of* $\dfrac{s+1}{(s-1)^2(s+2)}$

>> Let $\dfrac{s+1}{(s-1)^2(s+2)} = \dfrac{A}{s-1} + \dfrac{B}{(s-1)^2} + \dfrac{C}{s+2}$

or $s+1 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$

Put $s = 1: \; 2 = B(3) \quad \therefore \quad B = 2/3$

Put $s = -2: \; -1 = C(9) \quad \therefore \quad C = -1/9$

Equating the coefficient of s^2 on both sides we have,

$0 = A + C \quad \therefore \quad A = 1/9$

Now $L^{-1}\left[\dfrac{s+1}{(s-1)^2(s+2)}\right] = \dfrac{1}{9}L^{-1}\left[\dfrac{1}{s-1}\right] + \dfrac{2}{3}L^{-1}\left[\dfrac{1}{(s-1)^2}\right] - \dfrac{1}{9}L^{-1}\left[\dfrac{1}{s+2}\right]$

ie., $= \dfrac{1}{9}e^t + \dfrac{2}{3}e^t L^{-1}\left[\dfrac{1}{s^2}\right] - \dfrac{1}{9}e^{-2t}$

Thus $L^{-1}\left[\dfrac{s+1}{(s-1)^2(s+2)}\right] = \dfrac{1}{9}e^t + \dfrac{2}{3}e^t t - \dfrac{1}{9}e^{-2t}$

---

98. *Evaluate* $L^{-1}\left\{\dfrac{1}{s+3}+\dfrac{s+3}{s^2+6s+13}-\dfrac{1}{(s-2)^3}\right\}$

>> We have $L^{-1}\left[\dfrac{1}{s+3}\right]+L^{-1}\left[\dfrac{s+3}{s^2+6s+13}\right]-L^{-1}\left[\dfrac{1}{(s-2)^3}\right]$

$$L^{-1}\left[\frac{1}{s+3}\right]=e^{-3t}$$

$$L^{-1}\left[\frac{s+3}{s^2+6s+13}\right]=L^{-1}\left[\frac{s+3}{(s+3)^2+4}\right]=e^{-3t}\,L^{-1}\left[\frac{s}{s^2+2^2}\right]=e^{-3t}\cos 2t$$

$$L^{-1}\left[\frac{1}{(s-2)^3}\right]=e^{2t}L^{-1}\left[\frac{1}{s^3}\right]=e^{2t}\cdot\frac{t^2}{2!}=\frac{e^{2t}t^2}{2}$$

Thus the required inverse Laplace transform is given by

$$e^{-3t}+e^{-3t}\cos 2t-e^{2t}t^2/2$$

---

99. *Find the inverse Laplace transform of* (i) $\dfrac{2s-1}{s^2+2s+17}$ (ii) $\dfrac{e^{-2s}}{(s-3)^2}$

>> (i) $\dfrac{2s-1}{s^2+2s+17}=\dfrac{2s-1}{(s+1)^2+4^2}=\dfrac{2(s+1)-3}{(s+1)^2+4^2}$

$$L^{-1}\left\{\frac{2s-1}{s^2+2s+17}\right\}=L^{-1}\left\{\frac{2(s+1)-3}{(s+1)^2+4^2}\right\}=e^{-t}L^{-1}\left\{\frac{2s-3}{s^2+4^2}\right\}$$

$$=e^{-t}\left\{2\,L^{-1}\left(\frac{s}{s^2+4^2}\right)-3\,L^{-1}\left(\frac{1}{s^2+4^2}\right)\right\}$$

Thus $L^{-1}\left\{\dfrac{2s-1}{s^2+2s+17}\right\}=e^{-t}\left(2\cos 4t-\dfrac{3}{4}\sin 4t\right)$

(ii) $L^{-1}\left\{\dfrac{1}{(s-3)^2}\right\}=e^{3t}L^{-1}\left(\dfrac{1}{s^2}\right)=e^{3t}t=f(t)$ (*say*)

Now $L^{-1}\left\{\dfrac{e^{-2s}}{(s-3)^2}\right\}=f(t-2)\,u(t-2)$

Thus $L^{-1}\left\{\dfrac{e^{-2s}}{(s-3)^2}\right\}=\left\{e^{3(t-2)}\,(t-2)\right\}u(t-2)$

---

100. *Using convolution theorem, find the inverse Laplace transform of* $\dfrac{s}{(s-1)(s^2+4)}$

>> Let us write $\dfrac{s}{(s-1)(s^2+4)} = \dfrac{1}{s-1} \cdot \dfrac{s}{s^2+4}$

Now let $\bar{f}(s) = \dfrac{1}{s-1}$; $\bar{g}(s) = \dfrac{s}{s^2+4}$

$\therefore \quad L^{-1}[\bar{f}(s)] = f(t) = e^t$; $L^{-1}[\bar{g}(s)] = g(t) = \cos 2t$

We have convolution theorem,

$$L^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = \int_{u=0}^{t} f(u)\, g(t-u)\, du$$

ie., $$L^{-1}\left[\frac{s}{(s-1)(s^2+4)}\right] = \int_{u=0}^{t} e^u \cos(2t-2u)\, du$$

We have, $\displaystyle\int e^{ax} \cos(bx+c)\, dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)]$

Here $a = 1, b = -2, c = 2t, \ x = u$

$$L^{-1}\left[\frac{s}{(s-1)(s^2+4)}\right] = \left[\frac{e^u}{5}\left\{\cos(-2u+2t) - 2\sin(-2u+2t)\right\}\right]_{u=0}^{t}$$

Thus $$\mathbf{L^{-1}\left[\frac{s}{(s-1)(s^2+4)}\right] = \frac{1}{5}\left\{e^t - \cos 2t + 2\sin 2t\right\}}$$

---

101. *Find the inverse Laplace transform of* $\dfrac{1}{(s^2+4)(s^2+9)}$ *using convolution therorem.*

>> Let $\bar{f}(s) = \dfrac{1}{s^2+4}$; $\bar{g}(s) = \dfrac{1}{s^2+9}$

Taking inverse,

$$f(t) = \frac{\sin 2t}{2} \quad ; \quad g(t) = \frac{\sin 3t}{3}$$

We have convolution theorem,

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)\,g(t-u)\,du$$

$$\therefore \quad L^{-1}\left[\frac{1}{(s^2+4)(s^2+9)}\right] = \int_{u=0}^{t} \frac{\sin 2u}{2}\cdot\frac{\sin(3t-3u)}{3}\,du$$

$$\text{i.e.,} \quad = \frac{1}{6}\int_{u=0}^{t} \sin 2u\cdot\sin(3t-3u)\,du$$

$$= \frac{1}{6}\int_{u=0}^{t} \frac{1}{2}\left[\cos(2u-3t+3u)-\cos(2u+3t-3u)\right]du$$

$$= \frac{1}{12}\int_{u=0}^{t}\left[\cos(5u-3t)-\cos(-u+3t)\right]du$$

$$= \frac{1}{12}\left\{\left[\frac{\sin(5u-3t)}{5}\right]-\left[\frac{\sin(-u+3t)}{-1}\right]_{u=0}^{t}\right\}$$

$$= \frac{1}{12}\left\{\frac{1}{5}(\sin 2t+\sin 3t)+(\sin 2t-\sin 3t)\right\}$$

$$= \frac{1}{12}\left\{\frac{6}{5}\sin 2t-\frac{4}{5}\sin 3t\right\} = \frac{1}{10}\sin 2t-\frac{1}{15}\sin 3t$$

Thus $\boldsymbol{L^{-1}\left[\frac{1}{(s^2+4)(s^2+9)}\right] = \frac{1}{10}\sin 2t-\frac{1}{15}\sin 3t}$

---

103. *Solve the differential equation* $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y = -4\,;\; y(0) = 2,$ $y'(0) = 3$ *by using Laplace transforms.*

>> The given equation is $y''(x)+y'(x)-2y(x) = -4$

Taking Laplace transforms on both sides we have,

$$L[y''(x)]+L[y'(x)]-2L[y(x)] = L(-4)$$

$$\text{i.e.,} \quad \left\{s^2 Ly(x)-sy(0)-y'(0)\right\}+\left\{sL[y(x)]-y(0)\right\}-2L[y(x)] = \frac{-4}{s}$$

Using the given initial conditions we obtain,

$$(s^2+s-2)\,L[y(x)]-2s-3-2 = -\frac{4}{s}$$

ie.., $$(s^2+s-2)L[y(x)] = (2s+5)-\frac{4}{s}$$

ie., $$\{(s-1)(s+2)\}\,L[y(x)] = \frac{2s^2+5s-4}{s}$$

or $$L[y(x)] = \frac{2s^2+5s-4}{s(s-1)(s+2)}$$

$\therefore$ $$y(x) = L^{-1}\left[\frac{2s^2+5s-4}{s(s-1)(s+2)}\right]$$

Let $$\frac{2s^2+5s-4}{s(s-1)(s+2)} = \frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+2}$$

or $$2s^2+5s-4 = A(s-1)(s+2)+Bs(s+2)+Cs(s-1)$$

Put $s=0$: $-4 = A(-2)$ $\therefore$ $A=2$

Put $s=1$: $3 = B(3)$ $\therefore$ $B=1$

Put $s=-2$: $-6 = C(6)$ $\therefore$ $C=-1$

Hence we have,

$$L^{-1}\left[\frac{2s^2+5s-4}{s(s-1)(s+2)}\right] = 2L^{-1}\left[\frac{1}{s}\right]+L^{-1}\left[\frac{1}{s-1}\right]-L^{-1}\left[\frac{1}{s+2}\right]$$

Thus $\boldsymbol{y(x) = 2+e^x-e^{-2x}}$

---

103. *Using Laplace transforms solve* $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}-3y = \sin t$, *given* $y=0$, $\dfrac{dy}{dt}=0$ *when* $t=0$

>> The given equation is $y''(t)+2y'(t)-3y(t) = \sin t$ with the initial conditions $y(0)=0$ and $y'(0)=0$

Taking Laplace transforms as both sides of the given equation we have,

$$L[y''(t)]+2L[y'(t)]-3L[y(t)] = L(\sin t)$$

ie., $$\{s^2L[y(t)]-sy(0)-y'(0)\}+2\{sL[y(t)]-y(0)\}-3L[y(t)] = \frac{1}{s^2+1}$$

Using the given initial conditions wehave,

$$(s^2+2s-3)\,L[y(t)] = \frac{1}{s^2+1}$$

or $$L[y(t)] = \frac{1}{(s^2+1)(s^2+2s-3)} = \frac{1}{(s^2+1)(s-1)(s+3)}$$

$\therefore$ $$y(t) = L^{-1}\left[\frac{1}{(s^2+1)(s-1)(s+3)}\right]$$

Let $$\frac{1}{(s^2+1)(s-1)(s+3)} = \frac{As+B}{s^2+1}+\frac{C}{s-1}+\frac{D}{s+3}$$

or $$1 = (As+B)(s-1)(s+3)+C(s^2+1)(s+3)+D(s^2+1)(s-1)$$

Put $s = 1$: $1 = C(8)$ or $C = 1/8$

Put $s = -3$: $1 = D(-40)$ or $D = -1/40$

Put $s = 0$: $1 = B(-3)+C(3)+D(-1)$

ie., $1 = -3B+3/8+1/40$ or $B = -1/5$

Equating the coefficients of s^3 on both sides we get

$$0 = A+C+D \quad \text{or} \quad A = -1/10$$

$\therefore$ $$L^{-1}\left[\frac{1}{(s^2+1)(s-1)(s+3)}\right]$$

$$= \frac{-1}{10}L^{-1}\left[\frac{s}{s^2+1}\right]-\frac{1}{5}L^{-1}\left[\frac{1}{s^2+1}\right]+\frac{1}{8}L^{-1}\left[\frac{1}{s-1}\right]-\frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right]$$

Thus $$y(t) = \frac{-1}{10}\cos t-\frac{1}{5}\sin t+\frac{1}{8}e^t-\frac{1}{40}e^{-3t}$$

---

104. *Using Laplace transform method solve,* $\dfrac{d^3y}{dt^3}-3\dfrac{d^2y}{dt^2}+3\dfrac{dy}{dt}-y = t^2e^t$ *given* $y(0) = 1,\ y'(0) = 0,\ y''(0) = -2$

>> The given equation is

$$y'''(t)-3y''(t)+3y'(t)-y(t) = t^2e^t$$

Taking Laplace transform on both sides we have,

$$L[y'''(t)]-3L[y''(t)]+3L[y'(t)]-Ly(t) = L(e^t t^2)$$

ie., $\left\{s^3 L[y(t)] - s^2 y(0) - sy'(0) - y''(0)\right\} - 3\left\{s^2 L[y(t)] - sy(0) - y'(0)\right\}$
$+ 3\left\{sL[y(t)] - y(0)\right\} - L[y(t)] = \dfrac{2}{(s-1)^3}$

Using the given initial conditions we have,

ie., $(s^3 - 3s^2 + 3s - 1)\,L[y(t)] - s^2 + 2 + 3s - 3 = \dfrac{2}{(s-1)^3}$

ie., $(s-1)^3 L[y(t)] = (s^2 - 3s + 1) + \dfrac{2}{(s-1)^3}$

or $L[y(t)] = \dfrac{s^2 - 3s + 1}{(s-1)^3} + \dfrac{2}{(s-1)^6}$

$\therefore$ $y(t) = L^{-1}\left[\dfrac{s^2 - 3s + 1}{(s-1)^3}\right] + 2L^{-1}\left[\dfrac{1}{(s-1)^6}\right]$ . . . (i)

Now, $L^{-1}\left[\dfrac{s^2 - 3s + 1}{(s-1)^3}\right] = L^{-1}\left[\dfrac{\left\{(s-1)^2 + 2s - 1\right\} - 3s + 1}{(s-1)^3}\right]$

ie., $= L^{-1}\left[\dfrac{(s-1)^2 - s}{(s-1)^3}\right]$

$= L^{-1}\left[\dfrac{(s-1)^2 - (s-1) - 1}{(s-1)^3}\right]$

$= e^t L^{-1}\left[\dfrac{s^2 - s - 1}{s^3}\right]$

$= e^t\left\{L^{-1}\left[\dfrac{1}{s}\right] - L^{-1}\left[\dfrac{1}{s^2}\right] - L^{-1}\left[\dfrac{1}{s^3}\right]\right\}$

$\therefore$ $L^{-1}\left[\dfrac{s^2 - 3s + 1}{(s-1)^3}\right] = e^t\left(1 - t - \dfrac{t^2}{2}\right)$

Also $L^{-1}\left[\dfrac{2}{(s-1)^6}\right] = 2e^t L^{-1}\left[\dfrac{1}{s^6}\right] = 2e^t \dfrac{t^5}{5!} = \dfrac{e^t t^5}{60}$

Thus by using these results in the R.H.S of (i) we have,

$$y(t) = e^t\left\{1 - t - \frac{t^2}{2} + \frac{t^5}{60}\right\}$$

---

105. *Solve, using Laplace transform, the differential equation* $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 1 - e^{2x}$ *given that* $y(0) = 1$ *and* $\dfrac{dy}{dx} = 1$ *at* $x = 0$.

>> We have to solve $y''(x) - 3y'(x) + 2y(x) = 1 - e^{2x}$ subject to the conditions $y(0) = 1$ and $y'(0) = 1$

Taking Laplace transform on bothsides of the given equation we have,

$$L[y''(x)] - 3L[y'(x)] + 2L[y(x)] = L(1 - e^{2x})$$

i.e., $$\{s^2 L[y(x)] - sy(0) - y'(0)\} - 3\{s L[y(x)] - y(0)\} + 2L[y(x)] = \frac{1}{s} - \frac{1}{s-2}$$

Using the given initial conditions we have,

$$(s^2 - 3s + 2)\, L[y(x)] - s - 1 + 3 = \frac{-2}{s(s-2)}$$

ie., $$(s^2 - 3s + 2)\, L[y(x)] = (s-2) - \frac{2}{s(s-2)}$$

ie., $$(s-1)(s-2)\, L[y(x)] = (s-2) - \frac{2}{s(s-2)}$$

or $$L[y(x)] = \frac{1}{(s-1)} - \frac{2}{s(s-1)(s-2)^2}$$

$\therefore$ $$y(x) = L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{2}{s(s-1)(s-2)^2}\right] \quad \ldots(1)$$

Let $$\frac{2}{s(s-1)(s-2)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} + \frac{D}{(s-2)^2}$$

or $$2 = A(s-1)(s-2)^2 + Bs(s-2)^2 + Cs(s-1)(s-2) + Ds(s-1)$$

Put $s = 0$: $2 = A(-1)(4)$ $\therefore$ $A = -1/2$

Put $s = 1$: $2 = B(1)(1)$ $\therefore$ $B = 2$

Put $s = 2$: $2 = D(2)(1)$ $\therefore$ $D = 1$

Also by equating the coefficient of s^3 on both sides we get

$0 = A + B + C$ $\therefore$ $C = -3/2$

Now $L^{-1}\left[\dfrac{2}{s(s-1)(s-2)^2}\right]$

$$= -\frac{1}{2}L^{-1}\left(\frac{1}{s}\right) + 2L^{-1}\left(\frac{1}{s-1}\right) - \frac{3}{2}L^{-1}\left(\frac{1}{s-2}\right) + L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$= -\frac{1}{2}\cdot 1 + 2e^x - \frac{3}{2}e^{2x} + e^{2x}\cdot x$$

Hence (1) becomes

$$y(x) = e^x + \frac{1}{2} - 2e^x + \frac{3}{2}e^{2x} - e^{2x}\cdot x$$

Thus $\boldsymbol{y(x) = \frac{1}{2} - e^x + \frac{3}{2}e^{2x} - e^{2x}\cdot x}$ is the required solution.

8.5 Application of Laplace transforms

Vibrations of string

It is obvious that a spring vibrates when it is put into motion. Let us suppose that an elastic spring is suspended downwards and is appended with a body of mass m to its lower end. When the string carrying the mass vibrates freely there is a resistance due to the medium opposing the movement resulting in *damped vibrations*.

If y is the downward displacement of the body from the equilibrium position at time t then $\dfrac{dy}{dt}$ is the velocity, $\dfrac{d^2y}{dt^2}$ is the acceleration of the body at time t. y satisfies the differential equation

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0 \qquad \ldots (1)$$

where c is the damping coefficient and k is the spring modulus.

If the medium does not cause any resistance to the vibrations, then we have *undamped vibrations*.

In such a case $c = 0$ and (1) becomes

$$m\frac{d^2y}{dt^2} + ky = 0 \qquad \ldots (2)$$

If $c > 0$ then we have *resonance damped vibrations*

If an extra time dependent force function $f(t)$ is appended onto the body (1) and (2) assumes the form

$$m\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky = f(t) \qquad \ldots (3)$$

and $$m\frac{d^2y}{dt^2}+ky = f(t) \qquad \ldots (4)$$

Deflection of beams

Consider a uniform beam supported at both the ends is set to deflect from a horizontal position. Suppose it is subjected to a vertical load $w(x)$ then the deflection $y(x)$ at a distance x from one end of the beam satisfies the differential equation

$$EI\frac{d^4y}{dx^4} = w(x) \qquad \ldots (5)$$

where E is the modulus of elasticity and I is the moment of inertia. EI is a constant and is called flexural rigidity of the beam.

L-R-C Circuits

An electric circuit consisting of an inductance of L henrys, capacitance of C farads and resistance of R ohms connected in series is called an L-R-C circuit. If E volts is the e.m.f applied to an L-R-C circuit then the current i measured in amperes in the circuit at time t is given by the differential equation

$$L\frac{di}{dt}+Ri+\frac{q}{C} = E(t) \qquad \ldots (6)$$

Here q is the charge measured in coulombs. This is connected with the current i by the relation $i = \frac{dq}{dt}$

Thus (6) can also be put in the form

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C} = E(t) \qquad \ldots (7)$$

so that q can be found. Differentiating (6) we have

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{i}{C} = E'(t) \qquad \ldots (8)$$

so that i can be found.

The differential equation in an L-R circuit is given by

$$L\frac{di}{dt}+Ri = E(t) \qquad \ldots (9)$$

ILLUSTRATIVE PROBLEMS

1. *A particle moves along the x-axis according to the law* $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 25x = 0$. *If the particle is started at* $x = 0$ *with an initial velocity of 12ft/sec to the left, determine* x *in terms of t using Laplace transform method.*

>> The given equation is

$$x''(t) + 6x'(t) + 25x(t) = 0 \qquad \ldots (1)$$

and $x = 0$ at $t = 0$, $\frac{dx}{dt} = -12$ at $t = 0$, by data.

ie., $x(0) = 0$, $x'(0) = -12$ are the initial conditions.

Now taking Laplace transform on both sides of (1) we have

$$L[x''(t)] + 6L[x'(t)] + 25L[x(t)] = L(0)$$

ie., $\left\{s^2 L[x(t)] - sx(0) - x'(0)\right\} + 6\left\{sL[x(t)] - x(0)\right\} + 25L[x(t)] = 0$

Using the initial conditions we obtain,

$$(s^2 + 6s + 25)\ L[x(t)] = -12 \quad \text{or} \quad L[x(t)] = \frac{-12}{s^2 + 6s + 25}$$

$$\therefore \quad x(t) = -12L^{-1}\left[\frac{1}{s^2 + 6s + 25}\right]$$

$$= -12L^{-1}\left[\frac{1}{(s+3)^2 + 4^2}\right] = -12e^{-3t} L^{-1}\left(\frac{1}{s^2 + 4^2}\right)$$

ie., $x(t) = -12\, e^{-3t}\, \frac{\sin 4t}{4}$

Thus $\mathbf{x(t) = -3e^{-3t} \sin 4t}$

---

2. *Using Laplace transforms method solve the problem of resonance damped vibration of a spring.*

>> The governing d.e is given by $m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0\ ;\ c > 0$

or $$\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \frac{k}{m}y = 0$$

Let us denote $c/m = 2\lambda$ and $k/m = \mu^2$ for convenience so that the d.e assumes the form

$$y''(t) + 2\lambda y'(t) + \mu^2 y(t) = 0 \qquad \ldots (1)$$

Let $y(0) = y_0$ and $y'(0) = y_1$ be the initial conditions.

Taking Laplace transform on both sides of (1) we have,

$$L[y''(t)] + 2\lambda L[y'(t)] + \mu^2 L[y(t)] = 0$$

ie., $$\left\{s^2 L[y(t)] - s\,y(0) - y'(0)\right\} + 2\lambda\left\{sL[y(t)] - y(0)\right\} + \mu^2 L[y(t)] = 0$$

Using the initial conditions we obtain,

$$(s^2 + 2\lambda s + \mu^2)L[y(t)] - y_0 s - y_1 - 2\lambda y_0 = 0$$

ie., $$L[y(t)] = \frac{y_0 s}{s^2 + 2\lambda s + \mu^2} + \frac{(y_1 + 2\lambda y_0)}{(s^2 + 2\lambda s + \mu^2)}$$

$$y(t) = y_0 L^{-1}\left[\frac{s}{s^2 + 2\lambda s + \mu^2}\right] + L^{-1}\left[\frac{y_1 + 2\lambda y_0}{s^2 + 2\lambda s + \mu^2}\right]$$

$$= y_0 L^{-1}\left[\frac{(s+\lambda) - \lambda}{(s+\lambda)^2 + (\mu^2 - \lambda^2)}\right] + (y_1 + 2\lambda y_0)L^{-1}\left[\frac{1}{(s+\lambda)^2 + (\mu^2 - \lambda^2)}\right]$$

Denoting $\mu^2 - \lambda^2 = v^2$ we have,

$$y(t) = y_0 e^{-\lambda t} L^{-1}\left[\frac{s - \lambda}{s^2 + v^2}\right] + (y_1 + 2\lambda y_0)e^{-\lambda t} L^{-1}\left[\frac{1}{s^2 + v^2}\right]$$

$$y(t) = y_0 e^{-\lambda t}[\cos vt - (\lambda/v)\sin vt] + (y_1 + 2\lambda y_0)e^{-\lambda t}\sin vt/v$$

$$y(t) = e^{-\lambda t}[y_0 \cos vt + (\lambda y_0/v)\sin vt + y_1 \sin vt]$$

Thus $$\boldsymbol{y(t) = e^{-\lambda t}\left[y_0 \cos vt + \left\{y_1 + (\lambda y_0/v)\right\}\sin vt\right]}$$

where $\lambda = c/2m$, $\mu = \sqrt{k/m}$ and $v = \sqrt{\mu^2 - \lambda^2}$

---

3. *By using Laplace transforms, solve the problem of undamped forced vibrations of a spring in the case where the forcing function is* $f(t) = A\sin\omega t$

>> The differential equation associated with the problem is

$$m\frac{d^2y}{dt^2} + ky = A\sin\omega t \quad \text{or} \quad \frac{d^2y}{dt^2} + \frac{k}{m}y = \frac{A}{m}\sin\omega t.$$

Let $\lambda^2 = k/m$ and $\mu = A/m$

The d.e assumes the form

$$y''(t) + \lambda^2 y(t) = \mu\sin\omega t \qquad \ldots(1)$$

Let $y(0) = y_0$ and $y'(0) = y_1$ be the initial conditions.

Taking Laplace transform on both sides of (1) we have,

$$L[y''(t)]+\lambda^2 L[y(t)] = \mu L(\sin \omega t)$$

ie., $$\left\{s^2 L[y(t)]-s\,y(0)-y'(0)\right\}+\lambda^2 L[y(t)] = \frac{\mu\,\omega}{s^2+\omega^2}$$

Using the initial conditions we obtain,

ie., $$(s^2+\lambda^2)\; L[y(t)]-sy_0-y_1 = \frac{\mu\,\omega}{s^2+\omega^2}$$

ie., $$L[y(t)] = \frac{sy_0+y_1}{(s^2+\lambda^2)}+\frac{\mu\,\omega}{(s^2+\lambda^2)(s^2+\omega^2)}$$

$$\therefore \quad y(t) = y_0 L^{-1}\left[\frac{s}{s^2+\lambda^2}\right]+y_1 L^{-1}\left[\frac{1}{s^2+\lambda^2}\right]+\mu\,\omega L^{-1}\left[\frac{1}{(s^2+\lambda^2)(s^2+\omega^2)}\right]$$

$$y(t) = y_0 \cos\lambda\, t+\frac{y_1}{\lambda}\sin\lambda\, t+\mu\,\omega L^{-1}\left[\frac{1}{(s^2+\lambda^2)(s^2+\omega^2)}\right] \qquad \ldots (2)$$

In respect of the last term, let $s^2 = t$ for convenience and we resolve into partial fractions.

Let $$\frac{1}{(t+\lambda^2)(t+\omega^2)} = \frac{a}{t+\lambda^2}+\frac{b}{t+\omega^2} \quad \text{or} \quad 1 = a(t+\omega^2)+b(t+\lambda^2)$$

Put $t = -\omega^2$: $1 = b(\lambda^2-\omega^2)$ $\therefore$ $b = 1/\lambda^2-\omega^2$

Put $t = -\lambda^2$: $1 = a(\omega^2-\lambda^2)$ $\therefore$ $a = -1/\lambda^2-\omega^2$

$$L^{-1}\left[\frac{1}{(s^2+\lambda^2)(s^2+\omega^2)}\right] = \frac{1}{\lambda^2-\omega^2}\left\{L^{-1}\left(\frac{1}{s^2+\omega^2}\right)-L^{-1}\left(\frac{1}{s^2+\lambda^2}\right)\right\}$$

ie., $$L^{-1}\left[\frac{1}{(s^2+\lambda^2)(s^2+\omega^2)}\right] = \frac{1}{\lambda^2-\omega^2}\left(\frac{\sin\omega\, t}{\omega}-\frac{\sin\lambda\, t}{\lambda}\right) \qquad \ldots (3)$$

Thus by using (3) in the R.H.S of (2) we get,

$$y(t) = y_0 \cos\lambda\, t+\frac{y_1}{\lambda}\sin\lambda\, t+\frac{\mu}{\lambda^2-\omega^2}\left[\sin\omega\, t-\frac{\omega}{\lambda}\sin\lambda\, t\right]$$

---

4. *A particle undergoes forced vibrations according to the law* $x''(t)+25x(t) = 21\cos 2t$. *If the particle starts from rest at* $t = 0$ *find the displacement at any time* $t > 0$ *using Laplace transforms.*

>> We have $x''(t)+25x(t) = 21\cos 2t$; $x(0) = 0,\ x'(0) = 0$ from the given data.

Taking Laplace transform on both sides of the equation we have,

$$L[x''(t)]+25L[x(t)] = 21L(\cos 2t)$$

ie., $$\left\{s^2 L[x(t)]-s\,x(0)-x'(0)\right\}+25L[x(t)] = \frac{21s}{s^2+4}$$

Using the initial conditions we obtain,

$$(s^2+25)\ L[x(t)] = \frac{21s}{s^2+4}$$

$$\therefore \qquad x(t) = L^{-1}\left[\frac{21s}{(s^2+4)(s^2+25)}\right]$$

Let $$\frac{21s}{(s^2+4)(s^2+25)} = \frac{As+B}{s^2+4}+\frac{Cs+D}{s^2+25}$$

or $$21s = (As+B)(s^2+25)+(Cs+D)(s^2+4)$$

Comparing the coefficients of $s^3,\ s^2,\ s$ and constant on both sides we get,

$$A+C = 0 \quad ; \quad B+D = 0$$

$$25A+4C = 21 \quad ; \quad 25B+4D = 0$$

By solving these we get, $A = 1,\ B = 0,\ C = -1$ and $D = 0$

Hence $$L^{-1}\left[\frac{21s}{(s^2+4)(s^2+25)}\right] = L^{-1}\left[\frac{s}{s^2+4}\right]-L^{-1}\left[\frac{s}{s^2+25}\right]$$

Thus $\boldsymbol{x(t) = \cos 2t-\cos 5t}$

---

5. *A particle is moving with damped motion according to the law* $\dfrac{d^2x}{dt^2}+6\dfrac{dx}{dt}+25x = 0$. *If the initial position of the particle is at* $x = 20$ *and the initial speed is 10, find the displacement of the particle at any time t using Laplace transforms.*

>> The given equation is $x''(t)+6\,x'(t)+25x(t) = 0$

Initial conditions are $x(0) = 20,\ x'(0) = 10$

Taking Laplace transform on both sides of the equation we have,

$$L[x''(t)]+6\,L[x'(t)]+25L[x(t)] = 0$$

ie., $$\left\{s^2 L[x(t)]-sx(0)-x'(0)\right\}+6\left\{sL[x(t)]-x(0)\right\}+25L[x(t)] = 0$$

Using the initial conditions we obtain,

$$(s^2+6s+25)\ L[x(t)]-20s-10-120 = 0$$

ie., $L[x(t)] = \dfrac{20s + 130}{s^2 + 6s + 25}$

$\therefore$ $x(t) = L^{-1}\left[\dfrac{20s + 130}{(s+3)^2 + 16}\right]$

ie., $x(t) = L^{-1}\left[\dfrac{20(s+3) + 70}{(s+3)^2 + 4^2}\right]$

$= e^{-3t} L^{-1}\left[\dfrac{20s + 70}{s^2 + 4^2}\right] = e^{-3t}\left(20\cos 4t + \dfrac{70}{4}\sin 4t\right)$

Thus $\mathbf{x(t) = 10\,e^{-3t}(2\cos 4t + 7/4 \cdot \sin 4t)}$

---

6. *A voltage Ee^{-at} is applied at $t = 0$ to a circuit of inductance L resistance R. Show that the current at any time t is* $\dfrac{E}{R - aL}\left\{e^{-at} - e^{-Rt/L}\right\}$

>> The differential equation in respect of the L-R circuit is

$$L\frac{di}{dt} + Ri = E(t) \text{ where } E(t) = Ee^{-at} \text{ by data.}$$

Now the equation is put in the form

$$Li'(t) + Ri(t) = Ee^{-at}\ ;\ i(0) = 0$$

Taking Laplace transform (L_T) on both sides we get,

$$L\,L_T[i'(t)] + R\,L_T[i(t)] = E\,L_T(e^{-at})$$

ie., $L\left\{s\,L_T[i(t)] - i(0)\right\} + R\,L_T[i(t)] = \dfrac{E}{s+a}$

ie., $L_T[i(t)]\,(Ls + R) = \dfrac{E}{s+a}$ or $L_T[i(t)] = \dfrac{E}{(s+a)(Ls+R)}$

$\therefore$ $i(t) = L_T^{-1}\left[\dfrac{E}{(s+a)(Ls+R)}\right]$

Now, let $\dfrac{E}{(s+a)(Ls+R)} = \dfrac{A}{s+a} + \dfrac{B}{Ls+R}$

or $E = A(Ls+R) + B(s+a)$ ... (1)

Put $s = -a$: $E = A(-aL + R)$ $\therefore$ $A = \dfrac{E}{R - aL}$

Set $Ls + R = 0$ or $s = -R/L$ and from (1)

$$EL = B(-R + aL) \quad \therefore \quad B = \frac{-EL}{R - aL}$$

Now $\quad L_T^{-1}\left[\dfrac{E}{(s+a)(Ls+R)}\right] = \dfrac{E}{R-aL}\left\{L_T^{-1}\left[\dfrac{1}{s+a}\right] - L\,L_T^{-1}\left[\dfrac{1}{L(s+R/L)}\right]\right\}$

Thus $\quad \boldsymbol{i(t) = \dfrac{E}{R-aL}\left\{e^{-at} - e^{-Rt/L}\right\}}$

---

7. *The current i and charge q in a series circuit containing an inductance L, capacitance C, e.m.f E satisfy the D.E. $L\dfrac{di}{dt} + \dfrac{q}{C} = E$; $i = \dfrac{dq}{dt}$ Express i and q in terms of t given that L, C, E are constants and the value of i, q are both zero initially.*

>> Since $i = \dfrac{dq}{dt}$ the D.E becomes

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E \quad \text{or} \quad \frac{d^2q}{dt^2} + \frac{q}{LC} = \frac{E}{L}$$

ie., $\quad q''(t) + \lambda^2 q(t) = \mu$, where $\lambda^2 = 1/LC$ and $\mu = E/L$

Taking Laplace transform (L_T) on both sides we have,

$$L_T[q''(t)] + \lambda^2 L_T[q(t)] = L_T(\mu)$$

ie., $\quad \left\{s^2 L_T[q(t)] - sq(0) - q'(0)\right\} + \lambda^2 L_T[q(t)] = \dfrac{\mu}{s}$

But $\quad i = 0,\ q = 0$ at $t = 0$ by data.

That is $q(0) = 0,\ q'(0) = 0$

Hence $(s^2 + \lambda^2)L_T[q(t)] = \dfrac{\mu}{s} \quad$ or $\quad L_T[q(t)] = \dfrac{\mu}{s(s^2+\lambda^2)}$

$\therefore \quad q(t) = L_T^{-1}\left[\dfrac{\mu}{s(s^2+\lambda^2)}\right] \quad \ldots(1)$

Now $\quad \dfrac{1}{s(s^2+\lambda^2)} = \dfrac{1}{\lambda^2}\left(\dfrac{1}{s} - \dfrac{s}{s^2+\lambda^2}\right)$ by partial fractions.

$\therefore \quad L_T^{-1}\left[\dfrac{\mu}{s(s^2+\lambda^2))}\right] = \dfrac{\mu}{\lambda^2} L_T^{-1}\left[\dfrac{1}{s} - \dfrac{s}{s^2+\lambda^2}\right]$

ie., $\quad q(t) = \dfrac{\mu}{\lambda^2}(1 - \cos\lambda t)$ where $\lambda^2 = 1/LC$ and $\mu = E/L$

Thus $\quad \boldsymbol{q(t) = EC\left\{1 - \cos(\sqrt{1/LC}\,t)\right\}}$

---

8. *A resistance R in series with inductance L is connected with e.m.f $E(t)$. The current i is given by* $L\frac{di}{dt}+Ri = E(t)$. *If the switch is connected at $t=0$ and disconnected at $t=a$ find the current i in terms of t.*

>> We have by data $i=0$ at $t=0$ *ie.,* $i(0)=0$ and

$$E(t) = \begin{cases} E \text{ in } 0<t<a \\ 0 \text{ if } t \geq a \end{cases}$$

Also $Li'(t)+Ri(t) = E(t)$ is the given equation,

Taking Laplace transform (L_T) on both sides we have

$$L\,L_T[i'(t)]+RL_T[i(t)] = L_T[E(t)]$$

ie., $$L\left\{sL_T[i(t)]-i(0)\right\}+R\,L_T[i(t)] = L_T[E(t)]$$

ie., $$L_T[i(t)]\,(Ls+R) = L_T[E(t)] \quad \ldots(1)$$

Now to find $L_T[E(t)]$ we have by the definition,

$$L_T[E(t)] = \int_0^\infty e^{-st}E(t)\,dt = \int_0^a e^{-st}\cdot E\,dt + \int_a^\infty e^{-st}\cdot 0\,dt$$

$$= E\left[\frac{e^{-st}}{-s}\right]_0^a = \frac{-E}{s}(e^{-as}-1) = \frac{E}{s}(1-e^{-as})$$

Using this in the R.H.S of (1) we now have,

$$L_T[i(t)]\,(Ls+R) = \frac{E}{s}(1-e^{-as})$$

$$L_T[i(t)] = \frac{E(1-e^{-as})}{s(Ls+R)} = \frac{E}{s(Ls+R)} - \frac{Ee^{-as}}{s(Ls+R)}$$

$\therefore$ $$i(t) = L_T^{-1}\left[\frac{E}{s(Ls+R)}\right] - L_T^{-1}\left[\frac{Ee^{-as}}{s(Ls+R)}\right] \quad \ldots(2)$$

Now, let $\frac{E}{s(Ls+R)} = \frac{A}{s}+\frac{B}{Ls+R}$ by resolving into partial fractions.

or $E = A(Ls+R)+Bs$

Put $s=0$: $A = \frac{E}{R}$; Put $s = \frac{-R}{L}$: $E = B\left(\frac{-R}{L}\right)$ $\therefore$ $B = -\frac{EL}{R}$

Hence $$\frac{E}{s(Ls+R)} = \frac{E}{R}\cdot\frac{1}{s} - \frac{EL}{R}\cdot\frac{1}{Ls+R}$$

$$\therefore \quad L_T^{-1}\left[\frac{E}{s(Ls+R)}\right] = \frac{E}{R}L_T^{-1}\left(\frac{1}{s}\right) - \frac{E}{R}L_T^{-1}\left(\frac{1}{s+R/L}\right)$$

ie., $$L_T^{-1}\left[\frac{E}{s(Ls+R)}\right] = \frac{E}{R}(1-e^{-Rt/L}) \qquad \ldots(3)$$

Further we have the property of the unit step function,

$$L_T[f(t-a)u(t-a)] = e^{-as}\bar{f}(s) \text{ where } \bar{f}(s) = L_T[f(t)]$$

Taking $\bar{f}(s) = \dfrac{E}{s(Ls+R)}$ then $L_T^{-1}[\bar{f}(s)] = L_T^{-1}\left[\dfrac{E}{s(Ls+R)}\right]$

ie., $f(t) = \dfrac{E}{R}(1-e^{-Rt/L})$ by (3).

Also $L_T^{-1}[e^{-as}\bar{f}(s)] = f(t-a)u(t-a)$

ie., $$L_T^{-1}\left[e^{-as}\frac{E}{s(Ls+R)}\right] = \frac{E}{R}\left[1-e^{-R(t-a)/L}\right]u(t-a)$$

But $$u(t-a) = \begin{cases} 0 \text{ in } 0<t<a \\ 1 \text{ if } t \geq a \end{cases}$$

$$\therefore \quad L_T^{-1}\left[e^{-as}\frac{E}{s(Ls+R)}\right] = \frac{E}{R}\left[1-e^{-R(t-a)/L}\right] \text{ when } t \geq a \qquad \ldots(4)$$

$$= 0 \text{ in } 0<t<a$$

Using the results (3) and (4) in (2) we get

$$\mathbf{i(t) = \frac{E}{R}\left[1-e^{-Rt/L}\right] \text{ in } 0<t<a} \qquad \ldots(5)$$

Also, $$i(t) = \frac{E}{R}\left[1-e^{-Rt/L}\right] - \frac{E}{R}\left[1-e^{-R(t-a)/L}\right] \text{ when } t \geq a$$

ie., $$\mathbf{i(t) = \frac{E}{R}\left[e^{-R(t-a)/L} - e^{-Rt/L}\right] \text{ when } t \geq a} \qquad \ldots(6)$$

Thus (5) and (6) represents the required $i(t)$ in terms of t.

Remark : *Many of these application problems has also been solved in differential equations method (Refer Unit-III)*

- -

EXERCISES

Verify convolution theorem for the following functions. [1 to 4]

1. $f(t) = 1,\ g(t) = \cos t$
2. $f(t) = \sin at,\ g(t) = \cos bt$
3. $f(t) = t^2,\ g(t) = t e^{-2t}$
4. $f(t) = e^t,\quad g(t) = \cos t$

Applying convolution theorem find the inverse Laplace transform of the following functions. [5 to 10]

5. $\dfrac{s}{(s^2+4)^2}$
6. $\dfrac{s^2}{(s^2+9)^2}$
7. $\dfrac{1}{(s^2+1)^2}$
8. $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$
9. $\dfrac{1}{s^2(s^2+a^2)}$
10. $\dfrac{s+1}{(s^2+2s+2)^2}$
11. Show that $\int_0^t (t-u)^2 \cos 2u\, du = \dfrac{2}{s^2(s^2+4)}$
12. Show that $\int_0^t \sin(3t-3u)\, u^2 e^{-3u}\, du = \dfrac{6}{(s^2+9)(s+3)^2}$

Find $f(t)$ from the following integral equations. [13 to 16]

13. $f(t) = t + \int_0^t f(t-u)\, e^{-u}\, du$
14. $f(t) = 4t - 3\int_0^t f(u) \sin(t-u)\, du$
15. $f(t) = t + \frac{1}{6}\int_0^t f(u)(t-u)^3\, du$
16. $f'(t) = \int_0^t f(u)\cos(t-u)\, du\ ;\ f(0) = 1$

Solve the following differential equations by using Laplace transforms. [17 to 30]

17. $x''(t) - 2x'(t) + x = e^t\ ;\ x(0) = 2,\ x'(0) = -1$
18. $x''(t) + 4x'(t) + 4x(t) = 4e^{-2t}\ ;\ x(0) = -1,\ x'(0) = 4$
19. $y'''(t) + y'(t) = e^{2t}\ ;\ y(0) = 0 = y'(0) = y''(0)$
20. $y''(t) + y = 6\cos 2t\ ;\ y = 3,\ y' = 1$ at $t = 0$

21. $\frac{d^4 y}{dx^4} - 16y = 0$; $y = 1. y', y'', y'''$ are zero at $x = 0$

22. $y''(t) - y(t) = \cos h\, t,\ y(0) = 0 = y'(0)$

23. $y''(t) + \omega^2 y = a \cos(\omega t + \alpha)$; $y(0) = 0 = y'(0)$

24. $y''(t) + 9y = 18t$; $y(0) = 0,\ y(\pi/2) = 0$

25. $y''(t) + y = F(t)$ where $F(t) = \begin{cases} 0, & 0 < t < 1 \\ 2, & t > 1 \end{cases}$ and $y(0) = 0 = y'(0)$

26. $\frac{dx}{dt} + y = \sin t,\ \frac{dy}{dt} + x = \cos t$; $x = 2,\ y = 0$ when $t = 0$

27. $\frac{dx}{dt} + 2x + y = 0,\ x + \frac{dy}{dt} + 2y = 0$; $x(0) = 1,\ y(0) = 3$

28. $\frac{dx}{dt} + y = \sin t,\ \frac{dy}{dt} - x = e^t,\ x = 0,\ y = 1$ when $t = 0$

29. A particle moves along a line so that its displacement x from a fixed point at any time t is governed by the equation

$$\frac{d^2 x}{dt^2} + 4\frac{dx}{dt} + 5x = 80 \sin 5t$$

If the particle is initialy at rest, find the displacement at any time t.

30. In an L-R circuit a voltage $E \sin \omega t$ is applied at $t = 0$. If the current is zero initially find the current at any time.

ANSWERS

5. $\frac{t \sin 2t}{4}$

6. $\frac{1}{6}(3t \cos 3t + \sin 3t)$

7. $\frac{1}{2}(\sin t - t \cos t)$

8. $\frac{\cos bt - \cos at}{a^2 - b^2}$

9. $\frac{1}{a^3}(at - \sin at)$

10. $\frac{t e^{-t} \sin t}{2}$

13. $t + (t^2/2)$

14. $t + (3/2) \sin t$

15. $\frac{\sin t + \sin h\, t}{2}$

16. $1 + (t^2/2)$

17. $x(t) = e^t [2 - 3t + (t^2/2)]$

18. $x(t) = e^{-2t}(2t^2 + 2t - 1)$

2

19. $y(t) = \frac{-1}{2} + \frac{e^{2t}}{10} + \frac{2}{5}\cos t - \frac{1}{5}\sin t$

20. $y(t) = 5\cos t + \sin t - 2\cos 2t$

21. $y(x) = \frac{1}{2}(\cosh 2x + \cos 2x)$

22. $y(t) = \frac{t \sinh t}{2}$

23. $y(t) = \frac{a}{2\omega^2}\{\omega t \sin(\omega t + \alpha) - \sin\alpha \sin \omega t\}$

24. $y(t) = 2t + \pi \sin 3t$

25. $y(t) = 2\{1 - \cos(t-1)\} u(t-1)$

26. $x(t) = 2\cosh t, \quad y(t) = \sin t - 2\sinh t$

27. $x(t) = 2e^{-3t} - e^{-t}, \; y(t) = 2e^{-3t} + e^{-t}$

28. $x(t) = \frac{1}{2}(t\sin t + \cos t - e^t - \sin t), \quad y(t) = \frac{1}{2}(e^t + \cos t + 2\sin t - t\cos t)$

29. $x(t) = 2e^{-2t}(\cos t + 7\sin t) - 2(\cos 5t + \sin 5t)$

30. $i(t) = \frac{E}{L(\omega^2 + \lambda^2)}\left\{\omega e^{-\lambda t} + \lambda \sin \omega t - \omega \cos \omega t\right\}$ where $\lambda = R/L$.

BEATING THE MEMORY

[Formulae, Properties and Results to be remembered from all the units at a glance]

PART - A

Unit - I : Differential Equations - 1 [$p-y-x$ equations]

Equations solvable for $p = \dfrac{dy}{dx}$

A d.e of first order and n^{th} degree is of the form.

$$A_0 p^n + A_1 p^{n-1} + A_2 p^{n-2} + \cdots + A_n = 0 \qquad \ldots (1)$$

If this is expressible in the form

$$[p - f_1(x, y)]\,[p - f_2(x, y)] \ldots [p - f_n(x, y)] = 0$$

then we have n d.es of first order and first degree

$$[p - f_1(x, y)] = 0, [p - f_2(x, y)] = 0, \ldots [p - f_n(x, y)] = 0$$

If $\quad F_1(x, y, c) = 0,\ F_2(x, y, c) = 0, \ldots F_n(x, y, c) = 0$

are respectively the solution of these, then the product of all these represents the general solution of (1).

Equations solvable for y

$$y = f(x, p) \Rightarrow \text{d.e is solvable for } y.$$

Differentiate w.r.t x to obtain

$$\frac{dy}{dx} = p = f\left(x, y, \frac{dp}{dx}\right)$$

This being a d.e of first order in p and x the solution is of the form $\phi(x, p, c) = 0$.

Eliminating p from these two equations will give the general solution $G(x, y, c) = 0$.

Equations solvalbe for x

$$x = f(y, p) \Rightarrow \text{d.e is solvable for } x.$$

Differentiate w.r.t y to obtain

$$\frac{dx}{dy} = \frac{1}{p} = F\left(x, y, \frac{dp}{dy}\right)$$

This being a d.e of first order in p and y the solution is of the form $\phi(y, p, c) = 0$.

Eliminating p from these two equations will give the general solution $G(x, y, c) = 0$.

Singular solution

If $G(x, y, c) = 0$ is the general solution, treating c as a parameter we form $\frac{\partial}{\partial c}[G(x, y, c) = 0$

Elimination of c from these two equations will give $\phi(x, y) = 0$, known as the singular solution.

Clairaut's equation

This is of the form $y = px + f(p)$ whose general solution is $y = cx + f(c)$. Singular solution can also be obtained.

Unit - II : Differential Equations - 2

Solution of $f(D)y = \phi(x)$	
Stage-1	**Stage-2**
Finding C.F (y_c) by solving $f(D)y = 0$	Finding P.I (y_p) where $y_p = \phi(x)/f(D)$
Complete solution : y = C.F + P.I or $y = y_c + y_p$	

Stage-1 : Solution of $f(D)y = 0$

- Form the A.E $f(m) = 0$ and solve.
- C.F is based on the nature of the roots of the A.E as summarized in the following table.

	Nature of the roots of the A.E	Complimentary Function (C.F)
1.	$m_1, m_2, \ldots m_n$ (real and distinct)	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$
2.	$m_1 = m_2 = \cdots = m_n = m$ (n coincident real roots)	$(c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1}) e^{mx}$
3.	(a) $p \pm iq$ (a pair of complex roots) (b) $\pm iq$ (a pair of imaginary roots) (c) $p \pm iq$ repeated n times	$e^{px}(c_1 \cos qx + c_2 \sin qx)$ $c_1 \cos qx + c_2 \sin qx$ $e^{px}\{(c_1 + c_2 x + \cdots + c_n x^{n-1}) \cos qx + (c_1' + c_2' x + \cdots + c_n' x^{n-1}) \sin qx\}$

Stage-2 : Particular solution of $f(D)y = \phi(x)$ or $P \cdot I = y_p = \phi(x)/f(D)$

	Type of P.I	Method of getting the P.I
1.	$\frac{e^{ax}}{f(D)}$ (Also applicable for e^{ax+b}, $a^x = e^{\log a \cdot x}$, $\sinh ax$ $\cosh ax$ by using their definitions)	$\frac{e^{ax}}{f(a)}$, $f(a) \neq 0$; (D is replaced by a) $x \cdot \frac{e^{ax}}{f'(a)}$, $x^2 \cdot \frac{e^{ax}}{f''(a)}$ etc. [if $f(a) = 0$], [if $f'(a) = 0$]
2.	$\frac{\sin ax}{f(D^2)}$, $\frac{\cos ax}{f(D^2)}$ (Also applicable for $\sin(ax+b)$, $\cos(ax+b)$ & $\sin ax \cos bx$, $\sin ax \sin bx$ $\cos ax \cos bx$ by changing into sum)	$\frac{\sin ax}{f(-a^2)}$, $\frac{\cos ax}{f(-a^2)}$ where $f(-a^2) \neq 0$ (D^2 is replaced by $-a^2$) $x \cdot \frac{\sin ax}{f'(-a^2)}$, $x \cdot \frac{\cos ax}{f'(-a^2)}$ if $f(-a^2) = 0$ etc.
3.	$\frac{\phi(x)}{f(D)}$ where $\phi(x)$ is a polynomial in x.	Divide $\phi(x)$ by $f(D)$ by writing $\phi(x)$ in descending powers of x and $f(D)$ in ascending powers of D. Quotient = P.I as remainder will be zero.
4.	$\frac{e^{ax} V}{f(D)}$ where $V = V(x)$	$e^{ax} \cdot \frac{V}{f(D+a)}$ D is replaced by $D+a$ first & then find $V/f(D+a)$ premultiplied by e^{ax}
5.	$\frac{xV}{f(D)}$; $V = V(x)$ $\frac{x^2 V}{f(D)}$, $\frac{x^n \cos ax}{f(D)}$, $\frac{x^n \sin ax}{f(D)}$	$\left[x - \frac{f'(D)}{f(D)}\right] \frac{V}{f(D)}$ $x\left[\frac{xV}{f(D)}\right]$ (Repeated application), R.P. $\frac{e^{iax} x^n}{f(D)}$, I.P. $\frac{e^{iax} x^n}{f(D)}$

Unit - III : Differential Equations - 3

Method of variation of parameters

- Given $f(D)y = \phi(x)$ of the second order C.F is written in the form $y_c = c_1 y_1 + c_2 y_2$
- $y = A y_1 + B y_2$ is assumed to be the complete solution of the D.E where A, B are functions of x.
- y_1', y_2' and $W = y_1 y_2' - y_2 y_1'$ are found.

- Expressions for A' and B' is assumed in the form :
  $A' = \frac{-y_2\,\phi(x)}{W}$, $B' = \frac{y_1\,\phi(x)}{W}$ and simplified for the purpose of integration.
- Then $A = -\int \frac{y_2\,\phi(x)}{W}\,dx + k_1$, $B = \int \frac{y_1\,\phi(x)}{W}\,dx + k_2$
- $A(x)$, $B(x)$ are substituted in $y = A\,y_1 + B\,y_2$

Equations with variable coefficients reducible to equation with constant coefficients

Solution of Legendre's linear equation of second order in the form

$$a_0(ax+b)^2 y'' + a_1(ax+b)y' + a_2 y = \phi(x)$$

- Use the substitution $t = \log(ax+b)$ or $e^t = (ax+b)$. The following results are assumed to reduce the D.E into a D.E with constant coefficients.

$$(ax+b)y' = a\,Dy,\ (ax+b)^2 y'' = a^2 D(D-1)y \text{ etc. where } D = \frac{d}{dt}$$

Solution of Cauchy's linear equation of second order in the form

$$a_0 x^2 y'' + a_1 x y' + a_2 y = \phi(x)$$

- Use the substitution $t = \log x$ or $x = e^t$. The following results are assumed to reduce the D.E into a D.E with constant coefficients.

$$xy' = Dy,\ x^2 y'' = D(D-1)y \text{ etc., where } D = \frac{d}{dt}$$

- Linear differential equation with constant coefficients is solved to obtain $y = y_c + y_p$ in terms of t.
- Substitute for t to present the solution y in terms of x.

Series solution of differential equation

(i) Power series solution of a second order ODE :

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \qquad \ldots(1)$$

where $P_0(x) \neq 0$ at $x = 0$.

- Assume the series solution in the form $y = \sum_{r=0}^{\infty} a_r x^r$ and substitute into the given d.e.
- The coefficients of various powers of x are equated to zero.

- This results in $a_0, a_1 \neq 0; a_2, a_3, a_4 \ldots$ are obtained in terms of a_0 and a_1 and these are substituted in the assumed series solution.
- The required series solution is $y = a_0 F(x) + a_1 G(x)$, where $F(x)$ & $G(x)$ are infinite series.

(ii) Generalized power series method or Frobenius method.

The equation (1) will have $P_0(x) = 0$ at $x = 0$.

- Assume the series solution in the form $y = \sum_{r=0}^{\infty} a_r x^{k+r}$ and substitute into the d.e
- The coefficients of various power of x are equated to zero. The coefficient of the lowest degree term in x when equated to zero will result in a quadratic equation (*indicial equation*) in k giving two roots k_1, k_2.
- Further $a_0 \neq 0$ and $a_1, a_2, a_3 \ldots$ will be interms of a_0 only & these will be substituted in the assumed series solution. This will give $y = a_0 x^k F(x)$, $F(x)$ being an infinite series.
- Assuming that k_1, k_2 are real, distinct and donot differ by an integer $y_1 = a_0 x^{k_1} F(x)$ & $y_2 = a_0 x^{k_2} F(x)$ are two linearly independent solutions of (1).
- $y = Ay_1 + By_2$ constitutes the general solution of (1).

Unit - IV : Partial Differential Equations (PDE)

Formation of PDE

- Given a relationship of the form $f(x, y, z, a, b) = 0$, where $z = z(x, y)$ and a, b are arbitrary constants, the given relation is differentiated partially w.r.t x, y. Elimination of a, b from all these relations will give the PDE.
- If z is a relationship involved with two arbitrary functions, partial derivatives upto second order are found to eliminate the arbitrary functions for forming the PDE.

Solution of non homogeneous PDE by direct integration

In this method the dependent variable which being the solution is found by removing the differential operators through integration.

When integrated w.r.t x (*say*) a function of y is to be added, as a arbitrary constant and vice versa.

Solution of homogeneous PDE involving derivatives with respect to one independent variable only

The PDE is treated as an ODE and the solution is found first. The arbitrary constants in the solution are then replaced by arbitrary functions of the other variable giving the solution of PDE.

Solution of Lagrange's linear PDE : $Pp + Qq = R$

- Auxilarly equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ is formed first.
- Suitable pairs which can be put in the form
- $f(x)\,dx = g(y)\,dy,\ g(y)\,dy = h(z)\,dz,\ f(x)\,dx = h(z)\,dz$ are considered. Integration yields relations in (x, y) ; (y, z); (z, x)

or

- Multipliers k_1, k_2, k_3 ; k_1', k_2', k_3' are considered such that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{k_1\,dx + k_2\,dy + k_3\,dz}{k_1 P + k_2 Q + k_3 R} = \frac{k_1'\,dx + k_2'\,dy + k_3'\,dz}{k_1' P + k_2' Q + k_3' R}$$

- Integrating the two new expressions, two independent relations are obtained.
- Suppose $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are the two relations, $\phi(u, v) = 0$ constitutes the general solution of $Pp + Qq = R$

Method of separation of variables (Product method)

- Solution of the PDE is assumed in the form of a product.

  [$Ex : u = XY$ where $X = X(x), Y = Y(y)$]

- $u = XY$ is substituted into the given PDE to obtain equality of two ODEs.
- Equating each ODE to a common constant k, $X(x)$ and $Y(y)$ are obtained by solving the ODEs.
- Their product $XY = u$ represents the solution of the given PDE.

PART - B

Unit - V : Integral Calculus

Evaluation of double and triple integrals

- Double integral : $\int_{x=a}^{b} \int_{y=y_1(x)}^{y_2(x)} f(x, y)\,dy\,dx$

Double integral : $\displaystyle\int_{y=c}^{d}\int_{x=x_1(y)}^{x_2(y)} f(x, y)\,dx\,dy$

➢ Triple integral : $\displaystyle\int_{x=a}^{b}\int_{y=y_1(x)}^{y_2(x)}\int_{z=z_1(x,y)}^{z_2(x,y)} f(x, y, z)\,dz\,dy\,dx$

(*Similarly two other forms by cyclic rotation*)

Evaluation of $\iint_R f(x, y)\,dx\,dy$ over the specific region R

Draw the befitting figure from the given description to identify the specific region R. Then express I in the form (1) or (2) as follow.

$$I = \iint_R f(x, y)\,dx\,dy = \int_{x=a}^{b}\int_{y=y_1(x)}^{y_2(x)} f(x, y)\,dy\,dx \quad \ldots (1)$$

$$= \int_{y=c}^{d}\int_{x=x_1(y)}^{x_2(y)} f(x, y)\,dx\,dy \quad \ldots (2)$$

I is obtained by the evaluation of (1) or (2).

Evaluation of a double integral by changing the order of integration

Given the integral in either of the above form, say (1) identify the region of integration R by writing the figure and express (1) in the form (2). The evaluation of (2) will be the value of (1) on changing the order of integration. This can be vice versa also.

Evaluation of double integral by changing into polars

Given a double integral, use the substitution $x = r\cos\theta$, $y = r\sin\theta$ which will give $x^2 + y^2 = r^2$ and $dx\,dy = r\,dr\,d\theta$. Change the given limits of integration for x , y to r , θ suitably and evaluate.

Area, Volume and Surface area

1. $\iint_R dx\,dy =$ Area of the region R in the cartesian form.

2. $\iint_R r\,dr\,d\theta =$ Area of the region R in the polar form.

3. $\iiint_V dx\,dy\,dz =$ Volume of a solid.

4. If $z = f(x, y)$ be the equation of a surface S then the surface area is given by

$$\iint_A \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\; dx\,dy$$

where A is the region representing the projection of S on the xoy plane.

5. **Volume** of a solid (*in polars*) obtained by the revolution of a curve enclosing an area A about the initial line is given by

$$V = \iint_A 2\pi r^2 \sin\theta \, dr\, d\theta$$

Beta and Gamma functions

(i) $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \; ; \; m, n > 0$

(ii) $\beta(m, n) = 2 \int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \; d\theta$

(iii) $\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$

(iv) $\Gamma(n) = \int_0^\infty e^{-t}\, t^{n-1}\, dt, \; n > 0$

(v) $\Gamma(n) = 2 \int_0^\infty e^{-x^2} x^{2n-1}\, dx$

Properties and standard formulae / standard values

(i) $\beta(m, n) = \beta(n, m)$

(ii) $\Gamma(n) = (n-1)\,\Gamma(n-1)$ and $\Gamma(n) = (n-1)!$ if n is a positive integer.

(iii) $\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

(*Relationship between the beta and gamma functions*)

(iv) $\sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\,\Gamma(m)\,\Gamma(m+1/2)$ (*Duplication formula*)

(v) $\Gamma(m)\,\Gamma(1-m) = \pi/\sin m\pi$, where $0 < m < 1$

(vi) $\Gamma(1) = 1,\ \Gamma(1/2) = \sqrt{\pi},\ \Gamma(1/4)\,\Gamma(3/4) = \pi\sqrt{2},$

$\Gamma(1/3)\,\Gamma(2/3) = 2\pi/\sqrt{3},\ \Gamma(1/6)\,\Gamma(5/6) = 2\pi$

(vii) $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Unit - VI : Vector Integration

- If $\vec{F}(x, y, z)$ is a vector point function and C is any curve then $I = \int_C \vec{F}\cdot d\vec{r}$ is called the *vector line integral.*
- If $\vec{F}$ is the force acted upon by a particle in displacing it along the curve C then I represents the *total work done* by the force which also represents the *circulation* of $\vec{F}$ about C.
- If $\vec{F}$ is irrotational then the circulation of $\vec{F}$ is zero.

Green's theorem in a plane

If R is a closed region of the xoy plane bounded by a simple closed curve C and if M and N are two continuous functions of x, y having continuous first order partial derivatives in the region R then

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Stoke's theorem

If S is a surface bounded by a simple closed curve C and if $\vec{F}$ is any continuously differentiable vector function then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$$

Gauss divergence theorem

If V is the volume bounded by a surface S and $\vec{F}$ is a continuously differentiable vector function then

$$\iiint_V \text{div}\,\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,dS$$

where $\hat{n}$ is the positive unit vector outward drawn normal to S

Unit - VII : Laplace Transforms - 1

➢ $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$ is the *Laplace transform* of $f(t)$

➢ $L^{-1}[\bar{f}(s)] = f(t)$ is the *inverse Laplace transform.*

Table of Laplace transforms

	$f(t)$	$L[f(t)] = \bar{f}(s)$		$f(t)$	$L[f(t)] = \bar{f}(s)$
1.	a	$\frac{a}{s}$	5.	$\sinh at$	$\frac{a}{s^2 - a^2}$
2.	e^{at}	$\frac{1}{s-a}$	6.	$\sin at$	$\frac{a}{s^2 + a^2}$
3.	$\cosh at$	$\frac{s}{s^2 - a^2}$	7. (a)	t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$
4.	$\cos at$	$\frac{s}{s^2 + a^2}$	7. (b)	t^n $n = 1, 2, 3, ..$	$\frac{n!}{s^{n+1}}$

Properties of Laplace transforms

➢ If $L[f(t)] = \bar{f}(s)$ then

1. $L[e^{at} f(t)] = \bar{f}(s-a)$ 2. $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$

In particular, $L[t f(t)] = -\frac{d}{ds}\left[\bar{f}(s)\right]$, $L[t^2 f(t)] = \frac{d^2}{ds^2}[\bar{f}(s)]$

3. $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$ 4. $L\left[\int_0^t f(t)\,dt\right] = \frac{\bar{f}(s)}{s}$

Laplace transform of a periodic function $f(t)$ having period T

➢ $L[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,dt$

Unit step function (Heaviside function)

➢ $u(t-a)$ or $H(t-a) = \begin{cases} 0, & 0 < t \le a \\ 1, & t > a \end{cases}$